Mathematical Intuitionism and the Law of Excluded Middle

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Are there exceptions to the *law of excluded middle*? Is one of the first principles of human thought of only limited application? Why would one think that there are, or that it is?

The law became a matter of dispute among philosophers and mathematicians of the late nineteenth and early twentieth centuries, and it is in light of that dispute that I raise the question. I also wish to propose a relevant Thomistic distinction as being along the way of a solution.

The problem has to do with the foundations of arithmetic and, specifically, with a selective denial of the law of excluded middle. *Mathematical intuitionists*, who proposed this view, saw this denial as a necessary consequence of doing mathematics. *Classical mathematicians* in their turn defended the law as always and everywhere sound.

The law itself is variously stated. In modern logic, we say that $(p \vee \sim p)$ is always true.[1] Aristotle's own formulation is

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[1] We take $(p \vee \sim p)$ to mean that either a given proposition or its negation must be true, or that there is no third proposition between the two which is true. Anciently, the law of excluded middle was sometimes called *tertium non datur*, literally, "no third [alternative] is given." Also, one must make a distinction between the *negation* of modern logic and the *opposition* of Aristotelian logic. For both sorts of logic, the principle of excluded middle is taken to pertain to contradictories, though Aristotelians would maintain that propositions can be opposed in other ways as well. For example, it is not the case that one or the other of two contrary propositions must be true.

"there cannot be an intermediate between contradictories, but of one subject we must either affirm or deny any one predicate."[2] The point, of course, is that there can be no alternative beyond truly asserting a predicate of some subject or truly denying the same. If the law is sound, there could not possibly be a formulation for which neither of the contradictories is true.

What led the intuitionists to deny this? It is partly because they see intuitionism as a reaction to the classical way of looking at infinite collections of things. Up to that point, classical mathematicians held that all statements expressible in mathematical terms were either true or false,[3] including statements about infinite collections.[4] Yet, by the intuitionists' account, that assertion belied an assumption about the nature of the mathematical universe, namely, that there were such collections, in relation to which propositions made about them could be said to be true or false. Intuitionist mathematician Arend Heyting points out the assumption:

> You ought to consider what Brouwer's program was. It consisted in the investigation of mental mathematical construction as such, *without questions regarding the nature of the constructed objects, such as whether these objects exist independently of our knowledge of them.*[5]

[2] Aristotle, *Metaphysics*, trans. W. D. Ross (Oxford: Clarendon Press, 1924), 4.7.1011b23–5.

[3] To say the same thing with more precision, classical mathematicians held that either a statement expressible in mathematical terms, or its negation, is true.

[4] Illustrative discussions of the three major mathematical schools of modern times can be found in *Philosophy of Mathematics: Selected Readings*, ed. Paul Benacerraf and Hilary Putnam, (Cambridge: Cambridge University Press, 1987). In particular, the optimism of Hubert's so-called *programme* for mathematics was a target for the more conservative-minded intuitionists.

[5] Arend Heyting, "Disputation," in *Philosophy of Mathematics: Selected Readings*, 66. Emphasis mine. L. E. J. Brouwer was the founder of intuitionist mathematics.

The intuitionists' denial of the law of excluded middle is the consequence of a certain way of doing mathematics and of regarding the objects of mathematics. While classical mathematicians regard mathematical objects as existing independently of our knowledge of them, intuitionist mathematicians make no such claim. Heyting offers an example that he hopes will make his own position, as well as that of the classical mathematicians, clear. We paraphrase.

Is there a greatest pair of prime numbers that differ by one? The simple answer is: "yes." We can easily see whether the statement "some pair of prime numbers that differ by one is the greatest such pair" or its contradictory is true. Here the law of excluded middle applies: either the predicate "greatest" is truly affirmed of some pair of prime numbers differing by one or it is not—we also happen to know which is so.[6]

Is there a greatest pair of prime numbers that differ by two?[7] Here there is no simple answer. In this case intuitionists deny that the law of excluded middle holds: *neither the assertion nor the negation that there is such a pair is true*. Why the difference?

One can actually construct the greatest pair of primes that differ by one: for it is the number pair 2 and 3.[8] According to intuitionists, however, we only resolve the question when we have constructed the number pair that satisfies the condition given. Classical mathematicians, for their part, would say that the question is resolved already, as it were, within the infinite class of number called the integers. Given that we currently have no way to determine whether the number of twin primes is infinite or finite,[9] intuitionists conclude that

[6] Any prime number greater than 2 must be an odd number. Further, any odd number less 1 is an even number. Only the number pair 2 and 3, therefore, meet the conditions, since, among even numbers, only 2 is prime.

[7] Such pairs are called "twin primes."

[8] See note 6.

[9] All I mean by "infinite or finite" here is whether there is or is not a greatest. Classical mathematicians mean something decidedly different

the law of excluded middle does not apply in this case: neither the assertion that "some pair of twin primes is the greatest" nor its negation is true.

Heyting is well aware of the typical objection to his view:

> One may object that the extent of our knowledge about the existence or nonexistence of a last pair of twin primes is purely contingent and entirely irrelevant in questions of mathematical truth. Either an infinity of such pairs exist. . . or their number is finite: what does it matter whether we can actually calculate the number?[10]

The objection is reasonable and comes readily to mind. What have our ignorance of the right proof, and therefore our ignorance of the demonstrated truth or falsity of a given mathematical proposition, to do with the objective nature of the thing? Whatever truths humans have discovered to this day were also unknown to us at one time.[11] We surely would not claim that the assertion "magnetized bodies have fields that extend beyond their quantitative limits" was not true before it was discovered to be so. Realists, mathematical or otherwise, naturally maintain that our knowledge of a thing is measured by that thing and not vice-versa.[12] Being *unable* to determine whether there is an infinite number of twin primes is beside the point: for there either are or are not, in reality.

from this when they say "infinite or finite." Also, we know there are twin primes, (e.g., 5 and 7, 11 and 13, 17 and 19,) but have yet to discover a method for determining whether there is a greatest.

[10] Heyting, "Disputation," 67.

[11] *Pace* Plato.

[12] Cf. Aquinas's statement from his *Commentary on the Nicomachean Ethics:* "Natural philosophy examines the order of things that human reason considers but does not make, (we include both mathematics and metaphysics under natural philosophy)." St. Thomas Aquinas, *Sententia libri Ethicorum* (Rome: Leon. 1969), I.1, p. 4: "nam ad philosophiam naturalem pertinet considerare ordinem rerum quem ratio humana considerat sed not facit, ita quod sub naturali philosophia comprehendamus et mathematicam et metaphysicam."

Heyting responds by noting that this *realistic* view entails certain assumptions, in particular, assumptions about the existence of the objects in question:[13]

> Your argument is metaphysical in nature. If 'to exist' does not mean 'to be constructed', it must have some metaphysical meaning. It cannot be the task of mathematics to investigate this meaning or to decide whether it is tenable or not.[14]

Intuitionists prescind from the question whether mathematical objects exist outside the mind or not—to their credit, it is not a *mathematical* question. Their only concern is with those things they can construct mathematically. About all other questions they are silent.[15] Since classical mathematicians claim to know something about the greatest pair of twin primes (they claim to know that it is either there or not, which is something), they must assume that the infinite class of integers has a status independent of their thought of it—hence their metaphysical assumption.

The response of classical mathematicians to this objection is again quite predictable. Suppose the question whether there is an infinite number of twin primes were resolved. Suppose it had been demonstrated on January 1 of this year that there is no greatest pair of such numbers.[16] Would it not have been true to say, even last year, that there is an infinite number of twin primes? We would not have known, of course, but would the assertion not have been true all the same?

By the classical mathematicians account intuitionists fail to distinguish between the necessary and objective truth of

[13] To accuse classical mathematicians of hidden metaphysical assumptions was especially elegant. By and large, classical mathematicians have little regard for metaphysics.

[14] Heyting, "Disputation," 67.

[15] As *mathematicians*, of course; for the intuitionists are not doing mathematics in pointing out the flaws of classical mathematics.

[16] Bear in mind that the mathematicians previously mentioned would take this to mean that there is an infinite number of them.

a proposition and our (subjective) discovery of that truth, which is a contingent event. Intuitionists, in their turn, point out that the very distinction between the two presupposes a claim by the knowing mind concerning the existence of mathematical objects outside the mind knowing them. Thus Heyting:

> In the study of mental mathematical constructions, "to exist" must be synonymous with "to be constructed". . . . A mathematical assertion affirms the fact that a certain mathematical construction has been effected. It is clear that before the construction was made, it had not been made. Applying this remark to your example, we see that before January 1, . . . it had not been proved [that there is an infinite number of twin primes]. . . . But this is not what you mean. It seems that in order to clarify the sense of your question you must again refer to mathematical concepts: to some world of mathematical things existing independently of our knowledge, where [the statement, "there is an infinite number of twin primes"] . . . is true in some absolute sense.[17]

Intuitionists insist upon constructive proofs because they do not want to be forced to admit a world of mathematical objects existing independently of the mind. To be sure, mathematics has been associated with a sort of platonism in this regard: there are mathematical objects about which mathematicians claim to be able to form true propositions yet which evade any attempt at actual construction.[18] To return to the example, a classical mathematician would hold that there either is or is not an infinite number of twin primes, though we have not yet established which is so. The reality or nonreality of an infinite number of such pairs is presumed at the outset,

[17] Heyting, "Disputation," 67–68.

[18] Among the notable examples are the *transfinite numbers* of Russian-born philosopher and mathematician Georg Cantor, who was outspoken in his claim that such numbers have a separate existence outside the human mind. Yet no one claims to have *constructed* such numbers, at least as we understand that term here.

despite our inability to establish which. The matter is settled *in se*: all that remains is our discovery of it.[19]

For their part, intuitionists insist upon constructive proofs (and finite proof procedures[20]) precisely because they do not assume the existence of an extramental world of mathematical entities. Nor do they deny its existence. Their point is that mathematics is not metaphysics, that ought one not to rely upon a metaphysical assumption about the existence of infinite sets of numbers to do mathematics. For purposes of mathematics, the objects of mathematics exist only as we construct them: and we cannot say anything either in affirmation or denial of what does not (yet) exist. At the heart of the dispute, and the intuitionists' denial of the law of excluded middle, then, is a dispute over the existence of separate mathematical objects.[21]

There is a strong indeterminacy in the mind of intuitionists as they consider certain mathematical questions, even to the point of denying the law of excluded middle. This indeterminacy applies not only to the unresolved mathematical question (which is proper to the mind asking a question, for we do not know which of the alternatives is so), but even

[19] The classical mathematician's account accords well with the passage from Aquinas's *Commentary on the Nicomachean Ethics*, mentioned above. Intuitionist mathematics, on the other hand, is in direct contrast to Aquinas's claim there.

[20] "We find writers insisting, as though it were a restrictive condition, that in rigorous mathematics only a finite number of deductions are admissible in a proof—as if anyone had succeeded in making an infinite number of them." David Hilbert, "On the Infinite," in *Philosophy of Mathematics: Selected Readings*, 193.

[21] There is apparent precedent for a denial of the principle of excluded middle in the writings of Aristotle. Some take the discussion of the sea-battle in the *De Interpretatione* (9, 19a22–19b4) as his denial (at least) that a given proposition must be either actually true or actually false. Yet mathematicians would distinguish the sea-battle from the present instance: for mathematics deals with necessary things, not contingent ones. (Much more could be said concerning the *matter* of mathematics.)

to the object itself (for an object is neither one way nor the other before it exists, they say, and, as far as mathematicians are concerned, it exists only when it has been constructed). Classical mathematicians see indeterminacy only on the part of the mind of the one asking the question, not on the part of the object: *that* is already settled. The classical mathematician would say "I do not know whether there is a greatest pair of twin primes, nor may we ever be able to discover whether there is or is not such a pair: yet I do know that there either is or is not such a pair in reality." The dilemma is this: must we deny the law of excluded middle, even selectively, as do the intuitionists, or must we speak of the actual existence of an infinity of mathematical objects, as classical mathematicians would have us believe?[22] There is a middle position.[23] Resolving the problem as I propose entails saying that all mathematical propositions, or their contradictories, are true: there is no need to deny the law of excluded middle in mathematics. It also avoids distinct problems that arise when positing the existence of an actually infinite mathematical universe.[24] The solution entails seeing how *virtual* existence differs from actual and potential existence.[25]

[22] Remember that, for the classical mathematician, the statement "there is no greatest set of twin primes" means that there is an infinite number of such sets *actually in existence.*

[23] Luckily, the principle in question applies to the truth or falsity of a given proposition and its contradictory, not to just any conflicting opinions concerning arithmetical foundations.

[24] Cf. Aristotle, *Physics*, III, 4–8. See also Jean W. Rioux, "Cantor's Transfinite Numbers and Traditional Objections to Actual Infinity," *The Thomist* 64, no. 1 (January 2000): 101–25.

[25] Aquinas's notion of virtual existence does not of itself constitute a complete resolution to this problem; that would also entail an examination of assertions made about various objects in mathematics and, as mentioned, a consideration of the *matter* of mathematical demonstrations. Also to-the-point would be the question whether some mathematical assertions are statements about products of the human mind, *artifacts*, if you will, while others have an objective *scientific* character, much as

As virtual existence is something between actuality and mere potentiality, it must be distinguished from each in order to be properly understood. While the letter *De mixtione elementorum*[26] is primarily concerned with a problem other than the one before us, Thomas Aquinas uses the notion of virtual existence to resolve that problem, and so it is useful for us here. The question Aquinas presents in the *De mixtione* is: how are elements present in a compound?[27] It would seem that they are there neither actually nor merely potentially. On the one hand, they cannot be there actually, since the whole would then be a collection of elements, an aggregate, and not a "true mixture," a distinct substance. Thus Aquinas:

> So, the different parts of matter subsisting under the forms of the elements take on the notion of many bodies; but it is impossible for many bodies to be in a single place. So the four elements will not be in every part of the mixed body, and thus it will not be a true mixture (*vera mixtio*) but [only] according to sense, as happens with a collection of invisible or insensible bodies on account of their smallness.[28]

Aquinas would understand that term.

[26] St. Thomas Aquinas, "De mixtione elementorum ad Magistrum Philippum," (hereafter "*De mixtione*") in *Opuscula Philosophica*, (Turin: Marietti, 1954), 155–156. Apparently, this letter was anonymously appended to St. Thomas's unfinished *Sententia super libros De generatione et corruptione* as part of I.24.7. See James A. Weisheipl, *Friar Thomas D'Aquino: His Life, Thought, and Work* (Garden City: Doubleday, 1974), 395. In any case, that portion of the *Sententia* does not differ substantially from the *De mixtione*. For a parallel account see St. Thomas Aquinas, *Summa theologica* (hereafter, "*ST*") Ia 76.4 obj. 4 and ad 4.

[27] "Many have difficulty with how elements are in a mixture." Aquinas, *De mixtione*, #430: "Dubium apud multos esse solet quomodo elementa sunt in mixto." Aquinas is speaking here of a true mixture, as opposed to a mere aggregate of elements. See below.

[28] Aquinas, *De mixtione*, #431: "Diversae igitur partes materiae formis elementorum subsistentes plurium corporum rationem suscipiunt. Multa autem corpora impossibile est simul esse. Non igitur in qualibet parte corporis mixti erunt quatuor elementa; et sic non erit vera mixtio, sed secundum sensum, sicut accidit in congregatione invisibilium sive in-

On the other hand, the elements are not simply absent from the compound, or existing there in the manner in which all substances (due to the potentiality of prime matter) are in all other substances, namely, *merely* potentially. For a compound is composed of elements, and can be reduced to them:

> It seems to some that, since the active and passive qualities of the elements are brought to a mean in some way through alteration, the substantial forms must remain: for if they did not remain, it would seem to be a corruption of the elements, not a mixture.[29]

Further, it is of the nature of elements to be *in* the substances they compose:

> Further, if the substantial form of a mixed body is the act of matter that does not already have the forms of the simple bodies, then the simple bodies would lose the notion of elements. For an element is that out of which a thing is first made, *and which is in that thing*, and which is indivisible in species; for with the substantial forms [of the elements] lost, a mixed body would not be made up of simple bodies *such that they would remain in it*.[30]

The argument to this point is that elements must be present more than merely potentially (otherwise a compound would

sensibilium corporum propter parvitatem."

[29] Aquinas, *De mixtione*, #430: Videtur autem quibusdam quod, qualitatibus activis et passivis elementorum ad medium aliqualiter deductis per alternationem, formae substantiales elementorum maneant: si autem non remaneant, videtur esse corruptio quaedam elementorum, et non mixtio."

[30] Aquinas, *De mixtione*, #431: "Rursus, si forma substantialis corporis mixti sit actus materiae non praesuppositis formis simplicium corporum, tunc simplicia corpora elementorum amittent rationem. Est enim elementum ex quo componitur aliquid primo, et est in eo, et est indivisibile secundum speciem; sublatis enim formis substantialibus, non sic ex simplicibus corporibus corpus mixtum componetur, quod in eo remaneant." (Emphasis mine. The *Sententia super libros De generatione et corruptione* adds to this an internal reference to the definition of "element" given in *Metaphysics* 5.)

not be reducible to its elements) but not actually (since a compound is something more than a physical aggregate.) Aquinas's solution is that elements are *virtually* in compounds. "Therefore, the forms of the elements are not in a mixed body actually, but virtually."[31] Virtual existence is not the same as actual existence: what exists virtually does not exist in the primary sense of the word. Virtual existence is like potential existence in this. Still, it is not merely potential.[32] The potentiality of what exists virtually has been disposed, determined even, to one possible actuality among others. Using a phrase of Aquinas, the potentiality is already *determinatur ad unum*. In the *De veritate*, Aquinas maintains that the human intellect is like the potentiality of matter when it makes a judgment:

[31] Aquinas, *De mixtione*, #438–439: "Sunt igitur formae elementorum in mixtis non actu, sed virtute."

[32] According to Aquinas, since the qualities proper to the elements are present in the mean quality that is the proper disposition of the mixture, and since the qualities constituting that mean act *in virtue of* the substantial forms of the elements, the substantial forms of those elements must also be present in the mixture, and more than *merely* potentially. As he says: ". . . the active and passive qualities of the elements are contrary to each other, and admit of more and less. But from contrary qualities admitting of more and less a mean quality can be constituted . . . which is the proper quality of a mixed body . . . and this quality is the proper disposition to the form of a mixed body, just as a simple quality is to the form of a simple body. Therefore, just as the extremes are found in the mean, which participates in the nature of each, so the qualities of the simple bodies are found in the proper quality of a mixed body. Though the quality of a simple body is other than its substantial form, still, its acts in virtue of that substantial form (otherwise heat alone would heat). . . ." Aquinas, *De mixtione*, #438: ". . . qualitates activae et passivae elementorum sunt ad invicem contrariae, et suscipiunt magis et minus. Ex contrariis autem qualitatibus suscipientibus magis et minus constitui potest media qualitas . . . quae est propria qualitas corporis mixti, sicut qualitas simplex ad formam corporis simplicis. Sicut igitur extrema inveniuntur in medio, quod participat utriusque naturam, sic qualitates simplicium corporum inveniuntur in propria qualitate corporis mixti. Qualitas autem corporis simplicis est quidem aliud a forma substantiali ipsius, agit tamen in virtute formae substantialis, alioquin calor calefaceret tantum. . . ."

> Since, in itself, the possible intellect is in potency to all intelligible forms, just as prime matter is [in potency] to all sensible forms, the intellect is in itself no more determined to assent to the composition [affirmative judgment] than to the division [negative judgment], or conversely.[33]

There is nothing in the nature of the intellect that inclines it to one or the other of two contradictory assertions: it is in itself open to both. But what has no determination of its own (such as prime matter following the example from both *De mixtione elementorum* and *De veritate* or the intellect itself, as we see here in *De veritate,*) must be determined by something *ab extra*:

> Now, everything which is undetermined in relation to two things is only determined to one of them through something that moves it. But only two things move the possible intellect: its own object, which is an intelligible form or *quiddity* (as is said in *De anima III*), and the will, which moves all the other powers, as Anselm says. In this way, then, our possible intellect is related differently to the [two] parts of a contradiction.[34]

There are several possibilities regarding the *determinatio* of the intellect to one or the other contradictory. The first condition describes the current state of the mathematical instance we have been considering, twin primes:

[33] St. Thomas Aquinas, *Quaestiones Disputatae De veritatae* (hereafter "*De veritate*") (Turin: Marietti, 1949), *14.2.respondeo:* "Intellectus autem possibilis, cum, quantum sit de se, sit in potentia respectu omnium intelligibilium formarum, sicut et materia prima respectu omnium sensibilium formarum; est etiam, quantum est de se, non magis determinatus ad hoc quod adhaereat compositioni quam divisioni, vel e converso."

[34] Aquinas, *De veritate, 14.1.respondeo:* "Omne autem quod est determinatum [Leonine: *indeterminatum*] ad duo, non determinatur ad unum eorum nisi per aliquid movens ipsum. Intellectus autem possibilis non movetur nisi a duobus; scilicet a proprio obiecto, quod est forma intelligibilis, scilicet quod quid est, ut dicitur in III *De Anima*, et a voluntate, quae movet omnes alias vires, ut dicit Anselmus. Sic igitur intellectus noster possibilis respectu partium contradictionis se habet diversimode."

> For, sometimes it does not incline more to one than to the other, either because of a lack of moving principles, as in those problems for which we have no reasons [either way], or because of an apparent equality between the [reasons] for each side. This is the condition of one in doubt, who wavers between the two parts of a contradiction.[35]

There is no determination of the intellect in the case of twin primes, then, since we are unable to show one way or the other whether there is a greatest such pair. Among the four possible ways in which the intellect *can* be determined to one side of a contradiction, our concern is with the intellectual assent Aquinas calls "science" (*scientia*).[36]

Let us return to the question concerning how one thing can be in another virtually. Aquinas speaks in many places of things having principles as being virtually contained in these principles.[37] The example of the substantial forms of elements being virtually contained in compound bodies is one. He also points out, more to our purpose, that conclusions scientifi-

[35] Aquinas, *De veritate, 14.2.respondeo*: "Quandoque enim non inclinatur magid ad unum quam ad aliud, vel propter defectum moventium, sicut in illis problematibus de quibus rationes non habemus; vel propter apparentem aequalitatem eorum quae movent ad utramque partem. Et ista est dubitantis dispositio, qui fluctuat inter duas partes contradictionis."

[36] The other possible *determinations* of the intellect are opinion (an incomplete determination, accompanied by fear that the contradictory to what one holds is true,) understanding (a complete and immediate determination due to the intellect's apprehension of the truth of a proposition, that is, knowledge of first principles,) and belief (an unwavering determination effected by something sufficient to move the will, but not the intellect, of the one who believes.) Aquinas, *De veritate, 14.2.respondeo*.

[37] There are numerous citations. For the general principle, see St. Thomas Aquinas, *ST* Ia, 77.8, 93.7, II-IIae, 17.9 ad 3. More specifically, one can see the principle instantiated in 1) the relation between the intellectual soul and the sensitive and vegetative souls: *ST* Ia 76.3, 76.4 obj. 4 and ad 4; 2) the relation between effect and cause: *ST* Ia 4.2, I-IIae 20.5 obj. 1, II-IIae 88.5 ad 2; and, most importantly here, 3) the relation between conclusions known *scientifically* and the first principles: *ST* Ia 1.7, 94.3; I-IIae, 3.6.

cally known are virtually contained in their first principles. It is not difficult to see why this would be so. To say that conclusions known in this way are actually contained in their first principles would be to say that knowing the principles is the same as knowing the conclusions which follow from them, which is patently not so.[38] To say that the conclusions are merely potentially so contained would be to say that the principles might have been used to arrive at some other, even contradictory, conclusion, implying that in scientific demonstrations there is merely contingent matter (contingency being a simultaneous potency to opposites.)[39] Thus, the conclusions are *there*, that is, they are *determinately* contained in the first principles, but not actually.[40]

In a question concerned with how angels understand, St. Thomas elaborates upon this notion of virtual existence. His overall point is that angels do not understand things by *composition and division*, that is, by making affirmative or negative statements, but he makes distinctions on the way that bear directly upon the present question.

[38] Ample evidence is afforded the dubious reader from Descartes's *Geometry*, in which he stops short of explaining all the details of a particular geometrical problem ". . . because I should deprive you the pleasure of mastering it yourself. . . ," something more easily done by Descartes, apparently, than some of his readers. René Descartes, *The Geometry*, trans. David Eugene Smith and Marcia L. Katham (New York: Dover Publications: 1954), 10.

[39] Cf. Aristotle, *Metaphysics*, 5.5, 1015b6–8. See also Aquinas, *ST* Ia 86.3 and IIIa 3.5.

[40] To illustrate, one cannot immediately infer "all C is B" from "all A is B," nor can one deduce "all C is B" from "all C is A" *alone*. The truth of the conclusion is simply not to be found in either premise considered by itself. It is only when we conjoin the premises (through the act of discerning the middle term) that we can thereby see the truth of the conclusion. Yet it is not as though the two premises might have been used to deduce anything else. The conclusion "all C is B" (or its subalternate, "some C is B") is the only possible truth to which "all A is B" and "all C is A" can formally lead—there is no contingency here.

First, he finds a similarity between the intellect when it reasons and when it formulates judgments:

> Just as the conclusion is compared with the principle in the reasoning intellect, so is the predicate with the subject in the intellect composing and dividing. For if our intellect were immediately to see the truth of the conclusion in the principle, it would never understand by discursion or by reasoning. Similarly, if the intellect in apprehending the *quiddity* of the subject were immediately to grasp all that can be attributed to or removed from the subject, it would never understand by composing or dividing but only by understanding the *quiddity*. Clearly, then, it is for the same reason that our intellect understands by discursion as by composing and dividing: in first apprehending something, it cannot immediately see all that is virtually contained in it.[41]

Why is it that we do not see, in a single intellectual glance, all that the essence of a thing can contain, in *a priori* fashion, but must formulate propositions about the thing whereby we slowly come to understand better what it is? By Aquinas's account, our initial understanding of that essence is not so piercing as to enable us to see what is virtually there. In like fashion, if we were able to see at a glance all that can follow from the first principles, we would not need to deduce the conclusions separately. As Aquinas points out in another comparison, also in the *Summa*:

[41] Aquinas, *ST* (Turin: Marietti, n.d.) Ia 58.4 *respondeo:* "sicut in intellectu ratiocinante comparatur conclusio ad principium, ita in intellectu componente et dividente comparatur praedicatum ad subiectum. Si enim intellectus noster statim in ipso principio videret conclusionis veritatem, nunquam intelligeret discurrendo vel ratiocinando. Similiter si intellectus statim in apprehensione quidditatis subiecti haberet notitiam de omnibus quae possunt attribui subiecto, vel removeri ab eo; nunquam intelligeret componendo et dividendo, sed solum intelligendo quod quid est. Sic igitur patet quod ex eodem provenit quod intellectus noster intelligit discurrendo, et componendo, et dividendo; ex hoc scilicet quod non statim in prima apprehensione alicuius primi apprehensi potest inspicere quidquid in eo virtute continetur. . . ."

. . . the precepts are to the law what propositions are to the speculative sciences, in which the conclusions are virtually contained in the first principles. So, whoever perfectly and fully [*secundum totam suam virtutem*] understands the principles need not have the conclusions put to him later. But since not all who understand the principles are able to take into account everything virtually contained in them, it is necessary, for their sake, that conclusions be deduced from their principles in the sciences.[42]

Clearly, since Aquinas is speaking of scientific demonstration here, it is not a question of whether a given proposition can be demonstrated from first principles, but whether there is in principle a need to do so in order for it to be true; in other words, the question is not about whether some conclusion or its contradictory is true, but whether there is a need to demonstrate its truth, to explicitly connect it to the first indemonstrable principles. It would seem that Aquinas tends to side with classical mathematicians in this matter, for this is the sort of distinction they insist upon in their explanation of how a proposition can be true and yet not known to be true.

Returning to the question of twin primes, let us assert, hypothetically, that "no pair of twin primes is greatest." Classical mathematicians would claim that we spoke either truly or falsely, for they assume the preexistence of the infinite class of integers. Intuitionists would claim that what we said is neither true nor false before some construction has been made to establish what is the case.[43] What would Aquinas say?

[42] Aquinas, *ST* (Turin: Marietti, n.d.) II-IIae 44.2 *respondeo:* "hoc modo se habent praecepta in lege, sicut propositiones in scientiis speculativis; in quibus conclusiones virtute continentur in primis principiis. Unde qui perfecte cognosceret principia secundum totam suam virtutem, non opus haberet ut ei conclusiones seorsum proponerentur. Sed quia non omnes qui cognoscunt principia, sufficiunt considerare quidquid in principiis virtute continetur, necesse est propter eos ut in scientiis ex principiis conclusiones deducantur."

[43] Though it would seem that intuitionism would be unable to deal with negative propositions of this sort, (since they rely solely upon construction as proof in mathematics, and it is impossible to construct a negative, such as "nongreatest prime") nevertheless, they extend the word "construction" beyond what one might expect in mathematics. Some of them would regard an indirect argument, of the sort Euclid uses to prove that there is no greatest prime, as a type of construction. Thus, presumably, one would assume the construction of the greatest prime, merely to show that an absurdity results from it, thereby establishing the conclusion. See Stephan Korner, *The Philosophy of Mathematics: An Introduction* (New York: Harper & Brothers, 1960), 38.

He would say two things. First, he would uphold the law of excluded middle in this case also. Assuming that conclusions are virtually contained in first principles, Aquinas would claim, along with classical mathematicians, that either the statement or its contradictory is true even before it has been shown to be so by connecting it back to first principles (which is what demonstration is.) Any indeterminacy in the question is on the part of the mind asking it, not on the part of the thing itself. For what is virtually contained in first principles is determinately contained, unlike mere potentiality, in which there is a simultaneous potency to opposites (even contradictories), and therefore no necessity; in short, no conclusion.

On the other hand, if Aquinas would claim that either "no pair of twin primes is the greatest such pair" or its contradictory is true, would he also be required to accept the existence of an infinite class of integers outside the mind?

I do not believe so (and in this Aquinas agrees with the intuitionists.) For the truth of that proposition (or its contradictory, if it is falsified) is virtually contained in the principles of arithmetic; which is to say that the opposite could not be proved, or that the principles are already *determinatur ad unum* with respect to that specific disjunction. Though we cannot see all that is contained in the first principles of arithmetic, nevertheless what is contained therein is determinately so contained, in a manner sufficient also to determine the intellect and allow us to draw a conclusion to one of the two contradictory statements. Given the nature of the mathematical unit,[44] the property of being "greatest" is either compatible in part or universally incompatible with the subject "twin prime." Given our own frailty, as Aquinas has observed, given that we cannot see in a glance what the principles of arithmetic require in this instance, what remains is to discover the connections required to reach a conclusion (in this case, perhaps, whatever middle terms may be needed and the logical principle *modus tollens*), which, as we know, is a merely contingent event—it could occur now, or never.

[44] The mathematical unit is a principle and the first subject of arithmetic. See Thomas Aquinas, *Commentary on the Posterior Analytics*, 1.2 n. 17.