Definition in Geometry

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Introduction

The exposition of every science[1] presupposes a certain experience on the part of the learner in order that the instruction result, not in mere opinion, but in knowledge properly so called. Biology, for example, presupposes the experience of living things and their operations, and logic, that of discursive thought. Political science presupposes the vicarious experience of the origin, growth and decay of human societies acquired through the reading of history. Although such raw experience is a necessary propaedeutic to science, it is not sufficient. The first work of any science is to organize its proper experience by comparison and contrast. Once this is accomplished in the mind of the learner, the work of instructing him in the science itself can begin in earnest.

Metaphysics also presupposes a certain "experience," although not exactly in the same sense that the inferior sciences do. Its proper experience is having acquired a knowledge of those inferior sciences, or at least of their principles and methods.[2] The reason is that our first experience of 'being' is in

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[1] Unless otherwise indicated, the term *science* should be understood in the sense of *episteme* in Aristotle or *scientia* in St. Thomas Aquinas.

[2] Aristotle advises that the liberal arts be studied "only in a certain

those contracted states in which the inferior sciences study it: being *qua* mobile, being *qua* extended, and so on. Thus the first work of metaphysics is to organize its proper experience: finding the order among the inferior sciences. This entails observing the distinctions between the sciences and appreciating their principles and methods, both common and proper. The following thesis is a limited contribution to that endeavor, and thus should be judged neither as a work of mathematics proper nor of the "philosophy of mathematics," but of metaphysics.

The scope of this thesis is limited in three ways. First, it is limited to those sciences known as mathematical. Second, it is limited to one principle: definition. Third, it is limited to the mathematical development from Pythagoras to Descartes and Newton.

It is hoped that whatever light may be shed on these matters may illuminate the path to others of equal or greater importance. This hope is justified by the assumption that an investigation of mathematics is in some way prior to other possible investigations. I propose three reasons for thinking this to be the case.

First, mathematics has traditionally been placed first in the order of learning. The ability of the imagination to grasp its principles makes it proportionate to the mind of the beginner.[3] Furthermore, because of this proportionality, it engenders confidence in the learner about more difficult studies.[4] Likewise, the examination of the principle of definition in

degree" since the mastery associated with the expert precludes advancement to wisdom. Cf. *Politics*, Bk. VII, Chap. 2 (1337b 15–17) On the proper extent of the study of the inferior sciences, see Ernest L. Fortin, "The Paradoxes of Aristotle's Theory of Education," *Laval Théologique et Philosophique*, Vol. 13, no. 2, (1957) pp. 248–60.

[3] Cf. *In VI Ethicorum*, Lect. 7, n. 1209–11, *In Metaphysicorum Proemium; In Librum de Causis*, Lect. 1.

[4] Consider the eloquent testimony of Albert Einstein to this point cited on p. 70.

mathematics is a first step toward a more comprehensive study of definition generally.

Secondly, since the mode of abstraction in mathematics is peculiar among the sciences, its definitions require particular attention, lest we be haunted by the paradoxes of Platonic Forms.

Third, the method of mathematics, because of its rigor and orderly demonstration, is the paradigm for other disciplines, especially in modern times in which the mathematical aspect of natural science is often overemphasized. Thus the ability to judge whether a mathematical definition has been made well or badly will contribute much to our understanding of definition in those disciplines built upon a mathematical model.

Let us then outline the order of the present investigation. The following text from Aristotle suggests a natural place to begin:

> The framer of a definition should first place the object in its genus, and then append its differences; for of all the elements of the definition, the genus is usually supposed to be the principle mark of the essence of what is defined.[5]

Thus the first chapter will be devoted to the remote genus of all mathematical definition, namely *quantity*.

The division of the genus of quantity into its 'species',[6] *continuous* (magnitude) and *discrete* (multitude), provides the proximate genera for definition. But since these are studied by distinct and irreducible sciences (as will be shown), the modes of definition will differ. The second chapter will examine the relation between these sciences in some detail.

The third chapter begins the treatment of definition. Although the treatment applies to definition generally, mathematical examples have been employed wherever possible. The

[5] *Topics*, Bk. VI, Chap. 1 (139a 28–32); cf. *Posterior Analytics*, Bk. II, Chap. 15 (96b 15).

[6] As will become evident, magnitude and multitude are not species properly so-called, but rather meanings of an analogous name.

general principles of the third chapter will then be applied in the fourth chapter to cases peculiar to geometry, namely, the locus and the so-called "mechanical" definitions.

Finally, in order to illustrate the principles laid out in the preceding chapters, the various methods of defining the "conic sections" will be analyzed and compared. An epilogue will summarize the findings of the analysis.

I. Quantity as a Genus

1. The Place of Quantity among the Categories

Since we define a thing through its genus,[1] it will be necessary to inquire into the genus peculiar to mathematics, namely, quantity. Before investigating quantity itself, however, its relation to the other genera ought first to be established.

St. Thomas, in his *Commentary on the Physics*, distinguishes the categories through predication. Predication involves three principal ways according to which something can be said of a subject. According to the first, the predicate belongs to the very essence of the subject, as when man is said of Socrates or animal of man. The third, on the other hand, is completely extrinsic to the subject, and merely denominates it. But it is the second which is of interest here.

> Another mode is that in which what does not belong to the essence of a thing, but which inheres in it, is predicated of a thing. This is found either on the part of the matter of the subject, and thus is the predicament of *quantity* for quantity properly follows upon matter. . . , or else it follows upon the form, and thus is the predicament of *quality* (hence also, qualities are founded upon quantity, as color is in a surface, and figure is in lines or in surfaces), or else it is found in respect to another, and thus is the predicament of *relation*.[2]

[1] Not everything is definable through the genus, however. The supreme genera, precisely because they are supreme, are exceptions. While they cannot be defined in the strict sense, they can be distinguished from one another by certain marks which we will investigate in the case of quantity.

In this passage, St. Thomas associates quality with form and quantity with matter. But it is important to keep in mind that matter is always said *relative* to form. Thus the 'substantial form' is *form* with respect to prime matter. However, substantial form and prime matter, taken as a composite, constitute the *matter* with respect to the accidental forms. Hence, the first accidental form to inhere in substance is quantity. In turn, quality, or at least figure, which is the fourth 'species' of quality, inheres as its proper matter in quantity.[3] This gives rise to the ordering which St. Thomas presents elsewhere in his commentary:

> Among all the accidents which come to [*adveniunt*] substance quantity comes first, and then sensible qualities, and actions and passions [*passiones*], and the motion consequent upon sensible qualities.[4]

In the following sections, we will first consider the genus of quantity itself, and then what belongs to it.

Now something can belong to a genus in one of two ways.[5] The first way is by belonging to the genus properly. This can be either as an attribute of the genus, i.e., its marks or characteristics, or as a species of it. The second way is by belonging to the genus by reduction, as, for example, its determining principles. Each of these ways will be considered in its turn.

2. The Species of Quantity

It is significant that Aristotle, in treating quantity in the *Categories*,[6] immediately divides the genus into two species without

[2] *In III Physicorum*, Lect. 5, n. 322.
[3] A fuller account of these relationships will be given in Chapter III.
[4] *In II Phys.*, Lect. 3, n. 161.
[5] Cf. *Summa Theologiae* I, Q. 3, art. 5, corp.
[6] *Categories*, Chap. 6 (4b 20).

first indicating what is common to them, namely, the marks of the genus. He does not begin to designate these marks until midway through the chapter. The reason for this lies in Aristotle's general method of proceeding from the better known to the lesser known. The 'species' of quantity, magnitude and multitude, are far better known taken severally than as united under a genus.

A sign of this is that, in English,[7] we ask the question *How much?* of magnitude and may respond by *much* or *little*. Again, we ask *How many?* of multitude and respond with *many* or *few*. But we have no corresponding expressions for quantity in its generic meaning. Although the comparative form *more* is common to *much* and *many*, it is noteworthy that we intensify it by applying the adverbial forms of the latter two: *much more* and *many more*. Thus the original distinction is once more manifest.

Another sign is that we call the act of notifying multitude *counting* and that of magnitude *measuring*. That measuring involves counting does not disprove that they are distinct, but that counting is prior to measuring and more fundamental. Measurement presupposes counting, but counting does not presuppose measurement.[8] The term *reckoning* is often applicable to both but it is seldom used.

Thus, it is only by comparing magnitude and multitude as distinct that we come to appreciate them as 'species' under a common 'genus'[9] and distinguished by the 'specific differ-

[7] In English, at least, when properly spoken. The following anecdote brings to light a subtlety of the language often taken for granted by native speakers: "A woman in Goshen, Indiana, told me recently that her daughter-in-law, for whom English is a second language, was sitting on the floor one day in despair, surrounded by housekeeping items to be kept, put in the attic, or given away. 'So many junk!' she said." Newman, p. 8.

[8] I.e. multitude is logically *prior* to magnitude and thus quantity is not predicated of them equally; therefore they are not properly species of a genus.

[9] See footnote 6, p. 33, *supra*.

ences' of *continuous* and *discrete*. One of the strongest ties between these two is the common property of ratio. For we can readily see that a line can have to another line the same ratio that a number has to a number.[10] But the disparity between numbers and lines is reflected by the fact that the proportion cannot be alternated, i.e., a line has no ratio to a number. We are left with the mark that quantity is able to be divided into homogeneous, integral parts, of which the one is not the other.

The next division is that of magnitude into its 'species'.[11] If we follow Aristotle's division in the Greek text literally, as the medievals did,[12] we would say *line*, *surface* and *body*. But these should be understood in the more abstract senses of *length*, *area* and *volume*, for two reasons.

First of all, *body* and *surface* connote not only extension but *figure*, which belongs essentially to the genus of quality,[13] whereas *volume* and *area* connote only the supposit which figure terminates. Three signs of this can be taken from ordinary language: (a) We are inclined to say that we measure a thing's volume or area, rather than its body or surface. (b) We say that a body *has* volume and not that a volume *has* body. (c) We refer to a curved surface, but not a curved area. Again, line adds to length the notion of a limit or boundary.[14] An infinitely long object, if such could exist, would have length, but not *a* length, i.e., its length would not constitute a line.[15]

[10] *Elements*, Bk. X, Prop. 5.

[11] *Categories*, Chap. 6 (4b 24) This division is purely logical. Only volume is a magnitude in reality (*secundum rem*). Area and length *per se* have existence only in the mind (*secundum rationem*) as the determining principles of volume.

[12] Here are Aristotle's words and William of Moerbeke's translations:

a) *grammé*	a) *linea*
b) *epipháneia*	b) *superficies*
c) *sōma*	c) *corpus*

[13] This premise will be examined further in Sect. 4 below.

[14] *Metaphysics*, Bk. Δ, Chap. 13 (1020a 13).

[15] Cf. note 3, Chap. II, sect. 1.

Secondly, the mathematicians, who must be precise about their definitions, divide the genus into those magnitudes having length only, those having length and breadth, and those having length, breadth and depth.[16]

There is a paradox in Aristotle's division of magnitude which demands comment. In his *Metaphysics*,[17] he lists length, breadth and depth as species of magnitude, but adds speech, space and time to the list in the *Categories*.[18] Commenting on these passages, St. Thomas solves the riddle by observing that magnitude is considered in the latter insofar as it is a *measurement*; in the former, insofar as it is a *quantity*, strictly speaking:

> There [in the *Categories*] he distinguished between the species of quantity from the point of view of the different kinds of measure . . . whereas here [in the *Metaphysics*] he considers the species of quantity from the being of quantity.[19]

It remains to consider the marks of quantity.

3. *The Marks of Quantity*

The category of quantity has three marks or characteristics. The first is that quantities have no contraries. Aristotle, in the *Categories*, raises two objections to this which merit investigation here.

The first is that 'great' and 'small' refer to quantity and are contraries. Aristotle answers the objection in two ways. First, he shows that these are quantities considered as *related*, not as such, and hence belong properly to the category of relation. Second, he argues that 'great' and 'small' are not truly contraries, like 'black' and 'white,' but rather correlatives like 'master' and 'slave.' This is evident from the fact that things are always called 'great' or 'small' *with reference to another*. This

[16] Cf. *Elements*, Bk. I, Def. 1, "line"; Def. 5 "surface"; Bk. XI, Def. 1, "solid."

[17] *Metaphysics*, Bk. Δ, Chap. 13 (1020a 29–33).

[18] *Categories*, Chap. 6 (5a 6–14).

[19] *In Δ Meta.*, Lect. 15, n. 986.

is the mark of correlatives. Nevertheless, the point is confusing since 'great' and 'small' seem to admit of intermediates and thus resemble contraries.

Yet it is the second objection which is most interesting for our purposes. Earlier in his treatment of quantity, Aristotle noted that place (τόπος [*topos*]) can be considered as a quantity. But place seems to admit of contraries since "men define the term 'above' as the contrary of 'below.'"[20] This difficulty appears even more clearly in the *De Caelo*:

> But the two forms of rectilinear motion are opposed to one another by reason of their places; for up and down is a difference and a contrary opposition in place.[21]

Presumably, the same objection holds for the terms of any local motion.

After stating the objection, Aristotle does not give a separate answer, probably because it would so closely resemble his answer to the first. The contrariety involved in place does not belong to the placed thing *as quantified*, but follows upon some physical attribute or relation. The notion of direction is foreign to that of extension considered *as such*. We will return to this point later.

The second mark of quantity is that it does not admit of variation in degree. But this does not seem at first glance to be the case. Whatever varies in degree is susceptible to being more or less. And seven is certainly more than six and less than eight.

[20] *Categories*, Chap. 6, (6a 12). Dr. Sholz offers the following commentary on this text: "In order to understand this objection we must understand the ancient notion of the universe. The universe was said to be in the shape of a sphere, with the earth in the precise center. From this view of the universe, it seems to follow that the distance from the earth to the heavens is an ultimate or extreme distance, for one cannot go further from the center of the heavens, which contain all things under them and outside of which there is nothing." "The Category of Quantity," in *Laval Théologique et Philosophique*, Vol. 19, No. 2 (1963) p. 250.

[21] *De Caelo*, Bk. I, Chap. 4 (271a 4–6).

However, variation in degree is here contradistinguished from difference in kind. Thus while one shade may be more red than another, this seven apples is just as much seven as this seven cats. Quantities can only differ in kind. This second mark is a consequence of the first. Since quantity has no contraries, it has no true intermediates, and having no intermediates, it cannot have variation in degree.

The third mark is that equality and inequality are said of quantity. This mark is more proper and distinctive than the first two since it is not only true of every quantity, but true *only* of quantity.

The species and marks of quantity having been considered, it remains to consider its termination.

4. *The Termination of Quantity*

Based upon the order of the various categories laid out above, one might readily conclude that anyone studying quantity need not be concerned with quality, since the understanding of the former is prior to, and independent of, the latter. But it is important to see that, and in what sense, quantity does in fact depend on quality.

Both quantity and quality determine a subject, but in different ways. The medieval writers could simply manifest the distinction in Latin by the questions *quantum?* and *quale?*, to which quantity and quality are the genera of the respective answers. Now even though English, as we have seen above, has no question corresponding to *quantum?*, the expression 'What is it *like?*' does approximate the intent of the Latin *quale?*. A sign that the word 'like' connotes quality is this: If two things numerically distinct agree in substance, we say that they are 'the same', if in quantity, 'equal', if in quality, 'like.'[22]

Now, as long as we are considering, for example, that Socrates has size simply, we need not go beyond the categories of substance and quantity. Yet as soon as we say that Socrates has *a* size, we introduce the idea of a boundary or termination. "Having *a* size" connotes a kind of unity that "having size" does not. What determines size to be *this* size rather than another is its boundary, that beyond which it does not extend. Thus, having *a* size implies having a boundary, which for Socrates is his shape or figure. For the same reason, the geometer cannot speak only of area, but must speak about *an* area, namely an area bounded by a certain line or lines: this circle or that square. But since circle and square (and shape generally) are qualities properly belonging to quantities, they fall in the category of quality rather than quantity.[23]

Such shapes, however, are not considered by the geometer insofar as they are qualities, but only insofar as they are the *terminations* of quantity. For the geometer must consider the *principles* of magnitude as well as magnitude itself, even if those principles lie outside the proper subject (*genus subjectum*) of his science. The point *qua* indivisible, for example, is not itself a magnitude, but rather the termination of magnitude (length), and therefore a principle[24] of it. The geometer considers quality under this formality. The physician, on the other hand, would consider a shape such as the circularity of a wound insofar as it characterizes the wound. He thus knows something qualitative about it. The physician knows that circular wounds heal more slowly, the geometer knows why.[25]

Shape or figure is in some ways a principle of quantity, but in other ways an effect. If a given continuous quantity in two dimensions, for example, has a given number of boundaries whose lengths are in a certain proportion, and its perime-

[22] Cf. *Metaphysics*, Bk. Δ, Chap. 9, *passim.*

[23] In particular, the fourth 'species' of quality. Cf. Aristotle, *Categories*, Chap. 8 (10a 11). Again, this is not properly a species, but a meaning of an analogous name.

[24] In fact, this is the first meaning of "principle." Cf. Aristotle, *Metaphysics*, Bk. Δ, Chap. 1.

[25] *In I Post. Anal.*, Lect. 25, n. 212.

ter and area are given, its figure will be determined by these quantities.[26] On the other hand, figure itself determines certain properties, as whether the sum of the squares on the sides of some given triangle is greater than, less than, or equal to the square on the remaining side.[27] Nevertheless, figure is far better known as the principle of quantity than as its effect, and thus the geometer demonstrates through figure as if proceeding from a prior cause.

It should be noted that 'figure' is said of magnitudes and numbers in rather different senses. As applied to magnitude, it refers to the spatial disposition of its extended parts. This cannot be applied to numbers since they are distinguished from magnitude precisely because their parts are *not* disposed in such fashion. However, the parts of numbers are related to one another by equality or inequality, giving rise to Euclid's distinction between 'square' and 'cubic' numbers.[28] It is obvious, from the fact that these terms have been transposed from geometry, that 'figure' is used in arithmetic only analogously.

The terminations of quantity, or specifically, of magnitude, are divided into as many species as magnitude itself. Thus, length is terminated by points, area by lines, and volume by surfaces.

Now lines and surfaces, insofar as they are terminations of magnitude, also fall into the fourth 'species' of quality, since they can answer the question 'What is it like?'. To say that something is 'panduriform' (fiddle-shaped), for example, is to speak of it qualitatively rather than quantitatively. However, point does not, as such, answer the question 'What is it like?', and thus it is not included in the fourth 'species' of quality.

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[26] Cf. Courant and Robbins, *What Is Mathematics?*, pp. 231–32.

[27] *Elements*, Bk. I, Prop. 48; Bk. II, Prop. 12, 13.

[28] *Elements*, Bk. VII, Def. 18, 19.

II. Division of the Mathematical Sciences

1. *Number and Order of the Mathematical Sciences*

In examining the number and order of the mathematical sciences it will be advantageous to consider first the opinions of the writers of antiquity on this subject.

The first ordering, as Proclus relates, has come down to us from the school of Pythagoras:

> The Pythagoreans considered all mathematical science to be divided into four parts; one-half they marked off as concerned with [number][1] (ποσόν) [*poson*], the other half with magnitude (πηλίκον) [*pelikon*]: and each of these they posited as twofold. A [number] can be considered in regard to its character by itself or in relation to another [number], magnitudes as either stationary or in motion. Arithmetic, then, studies [number] as such; music, the relations between [numbers]; geometry, magnitude at rest; spherics,[2] magnitude inherently moving.[3]

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[1] Morrow incorrectly translates *poson* as "quantity" in this context. Cf. Heath, *Greek Mathematics*, Vol. I, p. 12.

[2] "Spheric," according to Heath, "means astronomy, being the geometry of the sphere considered solely with reference to the problem of accounting for the motions of the heavenly bodies." *Greek Mathematics*, Vol. I, p. 11.

[3] Proclus, *Commentary*, 35–36 (pp. 29–30). In light of the discussion of the termination of quantity in the previous chapter, it is interesting to note how Proclus concludes this paragraph:

> The Pythagoreans consider [number] and magnitude not in their generality, however, but only as finite in each case. For they say that the sciences study the finite *in abstraction from infinite quantities and magnitudes*, since it is impossible to comprehend infinity in either of them. Since this assertion is made by men who have reached the summit of wisdom, it is not for us to demand that we be taught about [number] in sense objects or magnitude that appears in bodies. To examine these matters is, I think, the province of the science of nature, not that of mathematics itself. (emphasis added).

This fourfold division appears to be the root of the *quadrivium* of the traditional liberal arts. We shall see later that it is exhaustive and irreducible, though not exactly as it is stated here. First, however, it will be useful to examine other enumerations which have been proposed by the writers of antiquity.

Plato, in his *Republic*, gives essentially the same list as Pythagoras, but counts plane geometry and solid geometry (στερεομετρία) [stereometria] as distinct sciences.[4]

Archytas, a contemporary and friend of Plato, follows the Pythagorean division:

> Thus they [the mathematicians] have handed down to us clear knowledge . . . about geometry, arithmetic and sphaeric, and last, not least, about music; for these μαθήματα [*mathemata*] seem to be sisters.[5]

Aristotle, in the *Posterior Analytics*,[6] gives a list of eight mathematical sciences: plane geometry, solid geometry, arithmetic, astronomy, optics, mechanics, harmonics and navigation. He divides them according to his distinction between knowledge of the fact (*quia*) and knowledge of the reasoned fact (*propter quid*). The first four sciences, he says, provide the *propter quid* of the truths contained in the last four, respectively.

Geminus, a Rhodesian Stoic, also lists eight sciences, but divides them two against six, as Proclus relates:

> But others, like Geminus, think that mathematics should be divided differently; they think of one part as concerned with intelligibles only and of another as working with perceptibles and in contact with them. By intelligibles, of course, they mean those objects that the soul arouses by herself and contemplates in separation from embodied forms. Of the mathematics that deals with intelligibles they posit arithmetic and geometry as the two primary and most authentic parts, while the mathematics that attends to sensibles contains six sciences: mechanics, astronomy, optics, geodesy, canonics and calculation.[7]

[4] *Republic*, Bk. VII, (528 b).

[5] Heath, *Greek Mathematics*, Vol. I, p. 11.

[6] *Posterior Analytics*, Bk. I, Chap. 13 (78b 33–79a 1).

Each of the foregoing accounts sheds light upon the various mathematical disciplines and the order among them. Yet none of these accounts is sufficient. For none of them gives either a unifying principle in virtue of which each of these sciences is called 'mathematical,' or the principles by which they can be clearly distinguished one from another. Such principles will be sought in the subsequent sections.

2. *Place of the Mathematical Sciences*

The theoretical sciences have traditionally been divided into physics, mathematics and metaphysics. In order to see what is common to the mathematical sciences, it will be helpful to examine the basis of this division.

At the beginning of his *Commentary on the Physics*, St. Thomas points out that the theoretical sciences should be distinguished according to their diverse modes of defining. He then shows that there are only three possible modes of defining since there are only three ways in which a definition is related to the thing defined:

> There are some things whose existence depends upon matter and which cannot be defined without matter. Further, there are other things which, even though they cannot exist except in sensible matter, have no sensible matter in their definitions. . . . There are still other things which do not depend on matter either according to their existence or according to their definitions.[8]

[7] Proclus, *Commentary*, 38 (p. 31) Cf. Heath, *Greek Mathematics*, Vol. I, p. 17: "It is the function of *geodesy* to measure, not a cylinder or cone (as such), but heaps as cones or pits as cylinders. *Canonic* is the theory of the musical intervals."

[8] *In I Physicorum*, Lect. 1, n. 2.

The third division is rather straightforward. These things in no way depend on matter. However, the third division does not bear on our problem. The mathematical sciences involve matter in some way and hence belong to either the first or second division. But the distinction between these is rather complicated, and we must proceed carefully. First, let us examine how St. Thomas explains the distinction here:

> These [the first two divisions] differ from one another as the curved differs from the snub. For the snub exists in *sensible matter* and it is necessary that *sensible matter* fall into its definition, for the snub is a curved nose. . . . But sensible matter does not fall into the definition of the curved, even though the curved cannot exist except in *sensible matter*. And this is true of all the mathematicals.[9]

Since it is not altogether clear from this passage what is meant by 'sensible matter', it will be helpful to examine a related text in which St. Thomas is more explicit:

> Mathematical species, however, can be abstracted by the intellect from sensible matter, . . . ; not from common intelligible matter, but only from individual matter. For *sensible matter is corporeal matter as subject to sensible qualities*, such as being cold or hot, hard or soft and the like: while *intelligible matter is substance as subject to quantity*.
>
> Now it is manifest that quantity is in substance before other sensible qualities are. Hence, *quantities*, such as number and dimension, and *figures*, which are the terminations of quantity, can be considered apart from sensible qualities; and this is to abstract them from sensible matter.[10]

In attempting to understand this passage, we should keep in mind two important points:

(1) As noted above, "matter" is always said *relatively* to some "form." The same thing can be viewed as matter in one respect and as form in another. Hence, whenever we speak of 'such-and-such matter' there must exist some corresponding 'such-and-such form' to which it refers. (The same principle does not apply to form which can exist without matter.)

[9] *Ibid.* (emphasis added).

[10] *Summa Theologiae*, I, Q. 85, art. 1, ad 2; cf. *In Boethium de Trinitate*, Q. 5, art. 3.

(2) We are considering here the various relationships existing among four distinct realities (*res*): *substance, quantity, figure* (the fourth 'species' of quality) and *sensible quality* (the third 'species' of quality). The rest (sensible matter, corporeal matter, intelligible matter, etc.) are distinctions *only in ratione*.

Quantity, then, can be considered as matter both with respect to figure and with respect to sensible quality. Thus quantity *qua* subject to figure is called *corporeal matter* and *qua* subject to sensible quality, it is called *sensible matter*. St. Thomas states the relationship between these ideas when he says that "sensible matter is corporeal matter as subject to sensible qualities."

If quantity is the matter of figure, then conversely, figure is the form of quantity. Thus figure *qua* inhering in quantity can be designated *corporeal form*. The composite of these is called *body* (Latin, *corpus*) and it is defined as what has a length, a depth and a breadth.[11]

In turn, substance[12] can be viewed as the matter of quantity. But since quantity can be related either to figure or to sensible quality, substance will also have a dual relationship. Substance, when considered as subject to figure, is called *intelligible matter*,[13] but as subject to sensible qualities it is *(common) sensible matter*. Examples of the latter are flesh and bone (*material parts of substance*) insofar as they are subject to hot and cold, etc.

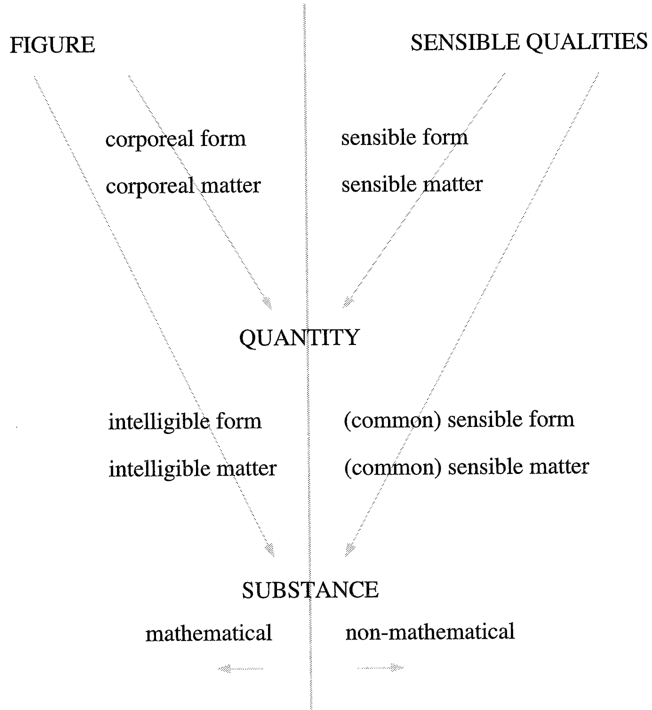
Clearly then, since the mathematician considers quantity and substance, not as related to sensible qualities, but only as

[11] *Summa Theologiae*, I, Q. 3, art. 1, obj. 1.

[12] I.e., the composite of prime matter and substantial form.

[13] This medieval term of art may be misleading to the modern reader. The "intellect" referred to here is, in the first instance, the imagination. *See* Deferrari, *Dictionary*, p. 632.

related to figure, he is said to abstract from sensible matter, but not from intelligible matter. (see figure)



We are now in a better position to understand still another text on the same topic, part of which has been quoted earlier:

> [The understanding of] the posterior does not belong to[14] the understanding of the prior, but conversely. Hence the prior can be understood without the posterior, but not conversely. . . .
>
> Among all the accidents which come to substance, quantity comes first, and then the sensible qualities, and actions and passions, and the motions consequent upon sensible qualities. Therefore quantity does not embrace in its intelligibility the sensible qualities or the passions or the motions. Yet it does include substance in its intelligibility. Therefore quantity can be understood without matter, which is subject to motion, and without sensible qualities, but not without substance. And thus quantities and those things which belong to them[14] are understood as abstracted from motion and sensible matter, but not from intelligible matter. . .
>
> Since, therefore, the objects of mathematics are abstracted from motion according to understanding and since they do not include in their intelligibility *sensible matter, which is the subject of motion*, the mathematician can abstract them from sensible matter.[15]

[14] Blackwell et al. consistently mistranslated "*non sunt de*" as "is not derived from"; pp. 78, 79.

It is evident from this text that quantity can be viewed as the subject not only of sensible qualities, as said above, but also of everything which is consequent upon the sensible qualities, including motion. Thus sensible matter is said to be the subject of motion.

The dual material role of quantity thus gives rise to the distinction between mathematics and natural philosophy. It is common to all the mathematical sciences to restrict the form to figure, either as it applies to magnitudes or multitudes. The diversity of these disciplines, however, remains to be explained.

3. The Primitive Mathematical Sciences

Two of the mathematical sciences, geometry and arithmetic, seem to be more fundamental than the others. Geminus recognized this, and divided the mathematical disciplines accordingly.[16] St. Thomas develops this point and manifests the hierarchy more explicitly:

> There are two kinds of sciences. There are some which proceed from a principle known by the natural light of the

[14] I.e. their terminations.
[15] *In II Physic.*, Lect. 3, n. 161 (emphasis added).
[16] See page 44 above.

intelligence, such as arithmetic and geometry and the like. There are others which proceed from principles known by the light of a higher science: thus the science of optics proceeds from principles established by geometry, and music from principles established by arithmetic.[17]

For this reason, geometry and arithmetic are often referred to as *primitive*; optics, music and the rest as *derivative*. The former will be considered here, and the latter in section 5 of this chapter.

Although Geminus held that geometry and arithmetic are primary, others, like Aristotle, posited the existence of a 'universal science of quantity',—presumably algebra, or something like algebra—which is prior to these:

> For one might raise the question whether first philosophy is universal, or deals with one genus, i.e. some one kind of being; for not even the mathematical sciences are all alike in this respect—geometry and astronomy deal with a certain particular kind of thing, while universal mathematics applies alike to all.[18]

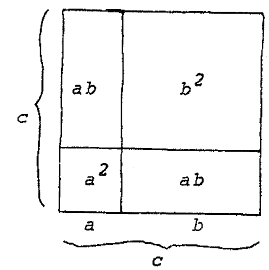
Aristotle goes on to argue that this branch of mathematics is prior precisely because it is more universal. But in order to see in what way universal mathematics is prior, we must first establish in what sense it is more universal.

A given algebraic equation may have either a geometrical or an arithmetic interpretation. In the equation:

$$(a + b)^2 = a^2 + 2ab + b^2$$

if *a* and *b* are taken to be two lines, the superscript to mean the square erected on a line and *ab* to be a rectangle, then this expression is equivalent to the following proposition from Euclid's *Elements*:

[17] *Summa Theologiae*, I, Q. 1, art. 2, corp.
[18] *Metaphysics*, Bk. E, Chap. 1 (1026a 23–27).



> If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.[19]

On the other hand, if *a* and *b* refer to numbers, the superscript to the operation of multiplying a number by itself, and *ab* to the product of *a* and *b*, the equation may be interpreted as follows:

> If the sum of any two numbers is multiplied by itself, the square number produced is equal to the sum of the two square numbers produced by multiplying each of the numbers by themselves plus twice the product of those two numbers.

Nevertheless, the algebraic equation cannot itself be the cause of our knowledge of either the geometrical or the arithmetic proposition. This is because the truth of the equation is only universal the way in which an analogous word is universal.[20] The symbols in the equation have no common univocal meaning. The only way to establish the universal truth of any algebraic expression is to present two independent proofs, one for numbers and one for lines.

Thus, with respect to our knowledge, algebra is neither prior to geometry and arithmetic, nor is it a mathematical

[19] *Elements*, Bk. II, Prop. 4.
[20] Cf. *Posterior Analytics*, Bk. I, Chap. 10 (76a 38).

science in the proper sense of the term. However, insofar as the propositions of algebra are more universal than those of the mathematical sciences, algebra is *prior* in the same sense that metaphysics or 'first philosophy' is prior to the inferior sciences.[21]

Assuming that there is no 'higher science,' that is, that there is no science higher than geometry and arithmetic, the question still remains whether geometry and arithmetic are irreducible. Proclus argues that the infinite divisibility of magnitude precludes such a reduction:

> If there were no infinity, all magnitudes would be commensurable and there would be nothing inexpressible or irrational, features that are thought to distinguish geometry from arithmetic.[22]

What Proclus means here is this. Suppose magnitudes were not infinitely divisible, but could be resolved into some minimal units. Then every line would contain a finite whole number of such units. That would mean that there would be some whole number ratio between any two given lines, as for example, between the side of a square and its diagonal. Thus, whatever relationship existed among magnitudes, they could be expressed by whole numbers and only one science would be necessary.

But some historians of mathematics maintain that Descartes succeeded in bridging the gap between these two sciences, although there is no consensus on whether he reduced arithmetic to geometry, geometry to arithmetic, or both to algebra. It will be illuminating to examine what exactly he did.

[21] For a systematic treatment of the nature of algebra, see E. V. Huntington's "The Fundamental Propositions of Algebra" in Young, pp. 150–97.

[22] Proclus, *Commentary*, Prologue, Part I, 6 (p. 5).

4. Cartesian Geometry

The first sentence of Descartes' *La Géométrie*[23] contains the assertion which the remainder of the work serves to justify, namely, that any construction required in geometry can, by his method, be restated in such a way that the construction depends solely on knowing the lengths of certain straight lines. His method, however, entails the introduction into geometry of operations heretofore undefined outside of arithmetic. He does so "in order to relate it [geometry] as closely as possible to numbers [arithmetic]."[24]

But Descartes was certainly not the first mathematician to exploit the analogy between numbers and lines. Euclid, who sharply distinguishes magnitude and multitude in the *Elements*, posits several axioms about addition and subtraction which apply to both 'species' of quantity.[25] Euclid even borrows properly geometrical terms such as 'square' and 'cube' and redefines them for arithmetic.[26] Nevertheless, Descartes' use of this general principle was original and has proved fruitful in solving certain classic problems.

Descartes observed that some of Euclid's purely numerical definitions could reasonably be extended to lines. Consider the definition of multiplication:

> A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other.[27]

[23] Some interpreters of Descartes have mistakenly identified the arithmetic operations of *La Géométrie* with those of his *Rules* (n. 8). It should be emphasized that "multiplication" and "division" of lines in the latter produce plane figures whereas in the former they produce other lines.

[24] *La Géométrie*, p. 297 (p. 2). [In all references to this text, numbers in parentheses refer to the English translation by Smith and Latham.]

[25] *Elements*, Bk. I, Common Notions 1–3.

[26] *Loc cit.*, Bk. I, Def. 22; Bk. VII, Def. 18, 19; Bk. XI, Def. 25.

[27] *Elements*, Bk. VII, Def. 15.

By modifying this definition, a number can be said to "multiply" a line when that line "is added to itself as many times as there are units" in the number. It would be meaningless, however, to speak of a line "multiplying" another line since the definition, as it stands, depends essentially on the supposition that one of the factors contains some number of units. To see how Descartes evades this difficulty we must return to the purely arithmetical operation of multiplication.

Whenever one number is multiplied by another to form some product, the same ratio is found between the unit and the multiplier which exists between the multiplicand and the product. If, for example, 3 multiplies 4 to make 12, then:

$$1 : 3 :: 4 : 12$$

Or generally,

unit : multiplier :: multiplicand : product

Thus we can state an alternative definition of multiplication:

> A number is said to multiply a number when a fourth proportional is found which is to one of the numbers as the other is to unity.

This definition eliminates the difficulty partially in that it does not presuppose that either factor contains units. But the definition is still not applicable to lines since it does presuppose some unit to which one of the factors has a ratio.

Descartes solves this last problem by "taking one line which I shall call unity . . . which ordinarily may be taken without restriction."[28] By this means, a definition of multiplication can be given for lines which "relates them so much the more to numbers"[29] by defining it by analogy with the equivalent definition above:

[28] *La Géométrie*, p. 297 (p. 2).
[29] *Ibid*.

> A line is said to multiply a line when a fourth proportional is found which is to one of the lines as the other is to 'unity.'

Or, to use algebraic notation, if there be lines *a* and *b* and 'unit' line *u*, then *ab* is the product when

$$u : a :: b : ab$$

In order to obtain a similar definition for the *division* of lines, one need only reverse the proportion as follows:

$$u : a/b :: b : a$$

To redefine the operation of *squaring* for lines, we begin with the Euclidean formulation:

> A square number is equal multiplied by equal.[30]

Its equivalent can be stated thus:

> A *square number* is the third proportional to a given number (its root) and unity.

This can be redefined for lines as:

> A square line is the third proportional to a given line (its root) and 'unity.'

In algebraic notation, letting *a* be any line, a^2 is its square when:

$$u : a :: a : a^2$$

The extraction of square roots can similarly be defined as finding the *mean* proportional.

Three differences between the arithmetical and geometrical operations are noteworthy:

First, while not every number is divisible by every other, nor does every number have a root, there are no such restrictions for lines.

Secondly, while the arithmetical unit is necessarily smaller than any given number, the geometrical "unit," since it is de-

[30] *Elements*, Bk. VII, Def. 18.

termined arbitrarily, that is, by convention, can be larger than either factors or roots and, *a fortiori*, larger than products or squares.

Finally, while no power higher than the "cube" is defined for numbers in Euclid, there is nothing to prevent the multiplication of lines by themselves indefinitely, thus making significant the expressions x^4, x^5, x^6, etc., which signify increasingly larger or smaller lines "in continued proportion."[31]

To those unfamiliar with Euclidean geometry, Descartes' use of an arbitrary unit may appear to be an innovation. But Descartes' unit is strikingly similar to Euclid's arbitrary "rational straight line" supposed throughout Book X of the *Elements*:

> Let, then, the assigned straight line be called *rational* and those straight lines which are commensurable with it, whether in length or in square, or in square only, rational, but those which are incommensurable with it, irrational.[32]

Even though no line in itself is rational or irrational, once the standard has been established, all lines can be designated as one or the other according as they are commensurable or not with the "rational straight line." Descartes merely takes this one step further by defining the particular ratios with the standard line.

It should now be apparent that Descartes neither reduced geometry to arithmetic, nor arithmetic to geometry. Nor does he invoke algebra except in borrowing some of its symbols to which he attaches a precise *geometrical* significance. In fact, Descartes, far from elevating algebra to the level of supreme mathematical science, referred to it as "a confused and obscure art which perplexes the mind."[33]

If then, geometry and arithmetic are the primitive mathematical sciences, we must now see how the derivative sciences are generated from them.

[31] Cf. *Elements*, Bk. VIII, Prop. 2.

[32] *Elements*, Bk. X, Def. 3.

[33] *Discourse on Method*, Part II, p. 550.

5. *The Derivative Sciences*

"Nature," as Heraclitus observed, "loves to hide." Hence, she cannot be investigated all at once. The philosopher of nature must consider one aspect of nature at a time, integrating his knowledge as he proceeds. The physicist, for example, begins to study the propagation of light by seeing how beams of light are reflected in a mirror or refracted in a lens.

But even this is too much to deal with at once. Even though light only travels in beams, the physicist must consider the beam, as it were, one ray at a time. Thus a visible 'solid'[34] in the sensible world is reduced, for the sake of simplicity, to a 'visible' line. This reduction having been made, the physicist can then employ all the knowledge offered by geometry about lines and the angles formed by them. In this way, a science called "optics" is generated which is neither purely natural nor purely mathematical.

St. Thomas describes such sciences as *scientiae mediae* or "middle" sciences:

> Those sciences are called *scientiae mediae* which take principles abstracted by the purely mathematical sciences and apply them to sensible matter. For example, optics applies to the visual line those things which are demonstrated by geometry about the abstracted line; and harmonics, that is, music, applies to sound those things which arithmetic considers about the proportions of numbers; and astronomy applies the consideration of geometry and arithmetic to the heaven and its parts.[35]

At first glance, the *scientiae mediae* would not appear at all fruitful. Mathematics abstracts from sensible qualities, while

[34] I.e., something with three dimensions; in this context, light is a 'solid'.

[35] *In II Physic.*, Lect. 3, n. 164.

the *scientiae mediae* merely put them back in. But the important point is that the sensible qualities are introduced only where the mathematical form renders them more intelligible. It is not fruitful for the physicist to add the qualities "bitter" and "sweet" to lines. But adding "visual" does in fact reveal truths about the observable behavior of reflected light in nature. Again, adding "visual" to numerical ratios does not help the musician to understand musical intervals, but adding "audible" does.

As we have seen above, sensible matter is called 'sensible' not only because it is subject to sensible qualities, but also because it is subject to all the accidents consequent upon sensible qualities. One of the most important of these is motion, which is added to the mathematicals in both astronomy and mechanics. St. Thomas accounts for this as follows:

> By its very nature, motion is not in the category of quantity, but it partakes somewhat of the nature of quantity from another source, namely, according as the division of motion derives from either the division of space or the division of the thing subject to motion. So it does not belong to the mathematician to treat of motion, although mathematical principles can be applied to motion. Therefore, inasmuch as the principles of quantity are applied to motion, the natural scientist treats of the division and continuity of motion, as is clear from the *Physics*.[36] And the measurements of motion are studied in the sciences intermediate between mathematics and natural science: for instance, in the science of the moved sphere and in astronomy.[37]

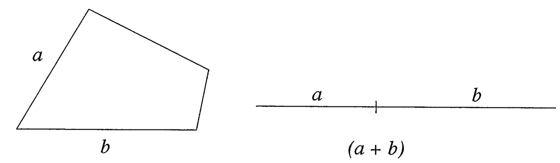
Furthermore, some sciences consider properties which follow upon motion without making reference to motion itself. An example is the study of directed magnitudes or *vectors*.[38]

[36] *Physics*, Bk. VI, Chap. 4 (234 b 21–235 b 5).

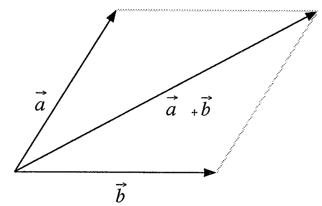
[37] *In Boethium de Trinitate*, Q. 5, art. 3, ad 5.

[38] This term is used here in the original geometric, rather than the modern algebraic sense: "At one time, vector quantities were defined as entities which involved magnitude and direction. The situation is now somewhat different. . . . Quantities which do not embody a magnitude or direction are, nevertheless regarded as 'vectors.'" Leaton, *Vectors*, p. 10.

In elementary geometry, if we are required to add two sides of a quadrilateral, we need not consider the relative position of the two sides.



If, however, these magnitudes are *directed*, say the measures of forces acting on a given body, then their sum must take the relative position of the lines into account.



The vector sum in this case would be the total directed force acting on the body. Such a sum is called a *resultant*. Although direction is extrinsic to magnitude as such, the addition of it is quite advantageous in the study of moving bodies.

We have noted that the *scientiae mediae* are neither purely mathematical nor purely natural. Yet it remains to determine to which branch of the theoretical sciences they belong most essentially.

6. Status of the Scientiae Mediae

In considering the status of the *scientiae mediae*, the Thomistic corpus does not seem to be consistent. In his *Commentary on the Physics*, St. Thomas agrees with Aristotle that they are closer to natural science:

> Although sciences of this sort are intermediate between natural science and mathematics, they are here said by the Philosopher to be more natural than mathematical, because each thing is named and takes its species from its terminus. Hence, since the consideration of these sciences is terminated in natural matter, then even though they proceed by mathematical principles, they are more natural than mathematical sciences.[39]

He goes on to say that:

> Astronomy is a natural science more than a mathematical one.[40]

St. Thomas even goes so far as to say that the *scientiae mediae* are *contrary* to mathematics, at least in the manner in which they are established:

> Sciences of this sort are established in a way contrary to the sciences which are purely mathematical, such as geometry or arithmetic. For geometry considers the line which has existence in sensible matter, which is the natural line. But it does not consider it insofar as it is in that sensible matter in virtue of which it is natural, but abstractedly, as was said.
>
> But optics, conversely, takes the abstracted line which is in the consideration of the mathematician and applies it to sensible matter and thus treats it not insofar as it is a mathematical but insofar as it is a physical thing.[41]

If optics, harmonics and the like are truly contrary to pure mathematics, we could hardly put them in the same branch of theoretical sciences. And since their subjects are defined with sensible matter, it seems clear that we should place them among the natural sciences.

However, in his *Commentary on Boethius' De Trinitate*, St. Thomas states just the opposite:

[39] *In II Physic.*, Lect. 3, n. 164.
[40] *Loc cit.*, n. 165.
[41] *Loc cit.*, n. 164.

> [Those sciences which] apply mathematical principles to natural things, for instance, music, astronomy and the like . . . have a closer affinity to mathematics.[42]

St. Thomas resolves this paradox by distinguishing the formal and material aspects of the subject of a *scientia media*:

> [In the subject of these sciences], that which is physical is, as it were, material, whereas that which is mathematical is, as it were, formal.[43]
>
> And since that which is formal is of most account, it follows that those sciences which draw conclusions about physical matter from mathematical principles, are reckoned rather among the mathematical sciences, though, as to their matter, they have more in common with physical sciences; and for this reason it is stated in the *Physics* (ii, 2) that they are more akin to physics.[44]

Recalling distinctions made above, we can state more precisely what this means. Quantity is the subject of both figure (*qua* corporeal matter) and of the sensible qualities (*qua* sensible matter). If quantity is allowed to take on one of those sensible qualities or their consequents, while still remaining material with respect to figure, the composite can be the subject of a *scientia media*.

In spite of being awkward ways to look at the natural world, these disciplines are powerful tools for the natural scientist who can successively subject the various sensible qualities, motions, etc., to mathematical forms, thereby illuminating certain truths which would otherwise have been obscure. Nevertheless, it is easy to see how the enterprise could mislead the unwary if the proper distinctions are not made.

It is now apparent that the mathematical sciences can be ordered by a fourfold division. On the one hand, we can distinguish between primitive and derivative sciences. On the

[42] *In Boethium de Trinitate*, Q. 5, art. 3, ad 6.
[43] *Ibid.*
[44] *Summa Theologiae*, II-II, Q. 9, art. 2, ad 3.

other, we can distinguish those sciences dealing with discrete quantity from those dealing with continuous quantity:

	CONTINUOUS	DISCRETE
PRIMITIVE:	GEOMETRY	ARITHMETIC
DERIVATIVE:	ASTRONOMY, ETC.	MUSIC

If astronomy were the only possible science derived from geometry, the Pythagorean division mentioned above would be exhaustive. However, optics, mechanics, etc., would also fall in this category. Why then were these not included in the *quadrivium*?

Perhaps the reason lies in the close association of these four disciplines with the order of traditional liberal education more than with their intrinsic character. The following explanation seems quite plausible:

> Astronomy and music are principal in that division, since geometry is ordered to the former, and arithmetic to the latter. Now it is reasonable to suggest that astronomy is a pre-figuration of the theoretical sciences generally, where knowledge is the end, since the stars are not things we can do something about—we can only learn about them. And the stars certainly seem to be, and were originally thought to be of a higher order than man, immortal and even divine. Furthermore, theoretical studies are ultimately concerned with the order of the universe as a whole, and it is from star-gazing and astronomy that we first begin to apprehend and wonder about that order. And the image of the astronomer, the man who doesn't see the things at his feet because he is looking up, forcefully suggests that liberal education concerns things higher than man.
>
> This is perhaps the reason why mechanics, as interesting as it is, is not one of the liberal arts. For in mechanics, geometry is applied to certain problems which are sub-lunar and on our own level, so to speak. Thus it does not express the fundamental orientation of the human mind, which is toward things better than man. Now whether the stars are really as the ancients supposed is not important for this argument; what is important is that they made astronomy rather than mechanics a liberal art.
>
> The science of music, on the other hand, would seem to pre-figure the practical or moral sciences, which concern the ordering of man's soul. For inasmuch as music imitates the passions of the soul, the discovery that arithmetical principles may be applied to musical tones suggests that a parallel order exists within the passions themselves. One is thereby led to suppose that the inclinations and affections can be ordered by reason, and that it is possible to understand how they ought to be ordered. However, the fact that the science of music is completely theoretical in mode also suggests that the basis of man's moral life is given by nature, rather than instituted by man himself.[45]

III. Definition in General

1. The Necessity of Definition

All theoretical science[1] begins with knowledge which is prior to science,[2] with concepts which are common to all men. These conceptions proceed from a natural inclination of the intellect to its object, antecedent to any deliberate intention to know. This is not to say that the *concepts* are innate, but rather that the disposition and power to know reality *through them* is natural.

Although the certitude of such concepts is unimpeachable, not all are equally distinct, as Plato observes in the *Phaedrus*:

> There are some [ideas] about which we all agree, and others about which we are at variance. . . . When someone utters the word 'iron' or 'silver' we all have the same object before our minds. . . . But what about the words 'just' and 'good'?

[45] Berquist, "Liberal Education and the Humanities," p. 5.

[1] I.e. *episteme* or *scientia*.

[2] Cf. *Posterior Analytics*, Bk. 1, Chap. I (71a 1–2).

Don't we diverge and dispute not only with one another but with our own selves?[3]

Such "disputable" concepts are apparently what Aristotle refers to in the opening paragraph of his *Physics*. He points out that "what is plain and obvious at first is rather what is *confused*."[4] St. Thomas, commenting on the passage, explains why such confusion is natural to the intellect:

> Those things are here called 'confused' which contain in themselves something potential and indistinct. And because to know something indistinctly is a mean between pure potency and perfect act, so it is that while our intellect proceeds from potency to act, it knows the confused before it knows the distinct. But it has complete science in act when it arrives, through resolution, at a distinct knowledge of the principles and elements. And this is the reason why the confused is known by us before the distinct.[5]

Thus the intellect can, by "disputing with itself," arrive at a more perfect understanding. But since this cannot be accomplished in a single act, several concepts must be composed into one whole. And this we call a *definition*.[6]

> For that which is defined is related to the things defining it as a kind of integral whole, insofar as the things defining it are in act in that which is defined. But he who apprehends a name, for example, *man* or *circle*, does not at once distinguish the defining principles. Whence it is that the name is, as it were, a sort of whole and is indistinct, whereas the definition divides into singulars, i.e., distinctly sets forth the principles of what is defined.[7]

It is clear then that a 'defining principle' is related to the thing defined as part to whole. But it is also related as prior to posterior, as Aristotle points out:

[3] *Phaedrus*, (263 a) p. 508.
[4] *Physics*, Bk. I, Chap. 1 (184a 22) (emphasis added).
[5] *In I Physic.*, Lect. 1, n. 7
[6] Greek: ὅρος [*horos*], lit. "boundary" or "landmark."
[7] *In I Physic.*, Lect. 1, n. 10.

> For the reason why the definition is rendered is to make known the term stated, and we make things known by taking not any random terms, but such as are *prior and more intelligible*. Accordingly, it is clear that a man who does not define through terms of this kind has not defined at all.[8]

Even mathematics, which of all the branches of theoretical science affords the most clarity, demands the subordination of posterior to prior among its concepts:

> It appears also in mathematics that . . . the most primary of the elementary principles are without exception very easy to show if the definitions involved e.g., the nature of a line or of a circle be laid down; . . . If on the other hand, the definitions of the starting points be not laid down, to show them is difficult and may even prove impossible.[9]

The preceding texts have been set forth to reveal how definition is related to the thing defined. We should next consider the relation of definition to science as a whole.

2. *Principles of Science*

Aristotle begins the *Posterior Analytics* by an examination of the first principles of discursive knowledge:

> All instruction given or received by way of argument proceeds from pre-existent knowledge. . . . The pre-existent knowledge required is of two kinds. In some cases admission of the fact must be assumed, in others, comprehension of the meaning of the term used, and sometimes both assumptions are essential.[10]

While Aristotle's observations apply to knowledge generally, we are here concerned with their application to demonstrative science.

[8] *Topics*, Bk. VI, Chap. 4 (141a 27–32) (emphasis added).
[9] *Op. cit.*, Bk. VII, Chap. 3 (158b 29, 36–39).
[10] *Posterior Analytics*, Bk. I, Chap. 1 (71a 1, 11–13).

After showing that the first principles must be undemonstrable and necessary, he distinguishes the various kinds of first principle. He first distinguishes between *proper* and *common*:

> Now of the premises used in demonstrative sciences, some are peculiar to each science and others common.[11]

Examples of common first principles are the "Common Notions" in Book I of Euclid's *Elements*, since they are applicable both to geometry and arithmetic. The proper principles are further divided into the *genus subjectum*, the definitions and the primary objects. He considers the *genus subjectum* first:

> The things peculiar to the science, the existence as well as the meaning of which must be assumed, are the things with reference to which the science investigates the essential attributes [i.e. the *genus subjectum*]. . . . With these things it is assumed that they exist and that they are of such and such a nature.[12]

For example, the geometer begins his study with the assumption that magnitude exists and that it is continuous quantity. Similarly, the arithmetician understands number to exist and to be discrete quantity. Such premises, however, need not be explicitly stated:

> There need not be any supposition as to the existence of the genus, if it is manifest that it exists.[13]

The next sort of principle is the definition, in which the essence is stated through a particular manifestation or attribute (*passio*) of the genus.[14] The definition neither affirms nor denies the existence of the thing defined, nor the inherence of

[11] *Op. cit.*, Bk. I, Chap. 10 (76a 37).
[12] *Loc cit.*, (76b 3–6).
[13] *Loc cit.*, (76b 17–18).
[14] Most properly, this would take the form of a specific difference. Less properly, it could be a commensurably universal property, accident, set of accidents or unique description. See sect. 4 *infra*.

the definition in any subject. All that is required is that the defined *passio* be understood.

Of the things defined, those whose existence is certain, yet undemonstrable, are the *primary objects*. Examples of these are the point, line and circle in elementary geometry and the unit in arithmetic. The existence of these objects is ascertained by postulates. Other objects, such as the triangle, are shown to exist by construction, i.e. by means of the primary objects.

It is worth noting how much emphasis Aristotle places on the fact that definitions neither affirm nor deny the existence of the things defined. Consider the following texts:

> (i) Arithmetic assumes the meaning of odd and even, "square" and "cube," geometry that of incommensurable, or of deflection or verging[15] of lines whereas the existence of these attributes is demonstrated.[16]
> (ii) Definitions . . . not hypotheses, for they do not assert the existence or non-existence of anything.[17]
> (iii) The definition of man and the fact that a man exists are different things.[18]
> (iv) What is meant by the word 'triangle' the geometer assumes, but [that it exists] he has to prove.[19]
> (v) Definition does not prove that the thing defined exists.[20]

Furthermore, we should examine how the doctrine presented here is actually put into practice in Euclid's *Elements*. For this leads us to a difficulty, the solution of which is rather illuminating.

[15] Cf. *Heath*, Euclid, Vol. 1, p. 150: Heath notes that neither "deflection" nor "verging" is "a geometrical figure, or an attribute of such a figure, or a part of a figure, but a technical term used to describe a certain problem. Euclid does not define such things."
[16] *Posterior Analytics*, Bk. I, Chap. 10 (76b 8–9).
[17] *Op. cit.*, Bk. I, Chap. 10 (76b 35–6).
[18] *Op. cit.*, Bk. II, Chap. 7 (92b 10).
[19] *Loc cit.*, Bk. II, Chap. 7 (92b 16–17).
[20] *Loc cit.*, Bk. II, Chap. 7 (92b 19).

Let us take as an example the notion of "parallel." Euclid lays out the definition at the beginning of Book I:

> Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.[21]

In a series of propositions in Book I, Euclid proves a number of things about parallel lines. Propositions 27 and 28 show under which conditions lines are parallel. In both cases, the lines are supposed cut by a transversal. The former shows that the equality of the alternate angles implies parallels. The latter shows two other conditions which imply them: the equality of an exterior angle with an interior opposite one, and the equality of the interior angles with two right angles. Thus, in the two propositions, three sufficient conditions for parallels are shown. Proposition 29 then proves that these three conditions are also necessary, i.e., they are necessary consequents of parallel lines being cut by a transversal. Proposition 30 shows that straight lines parallel to the same straight line are parallel to one another.

What is curious here is that parallel lines have not heretofore been proved to exist. Euclid does not construct them until Proposition 31. It appears that the four previous propositions must be merely hypothetical. One would think that Euclid, in the interests of scientific rigor, should have constructed parallel lines first using only the definition stated above.

The paradox is resolved, however, when the method of proving existence is more carefully examined, as St. Thomas explains:

> When the existence of a cause is demonstrated from an effect, this effect takes the place of the definition of the cause in proof of the cause's existence. . . .[22]

Let us see how this applies in the case at hand.

[21] *Elements*, Bk. I, Def. 23.

[22] *Summa Theologiae*, I, Q. 2, art. 2, ad 2.

Euclid shows in Propositions 27 and 29 that a formal effect of parallel lines, namely, the equality of the alternate angles, is commensurately universal with the parallel lines, its cause. When, in Proposition 31, he constructs two lines which exhibit this effect, he can show immediately that the cause is present, i.e., that the lines are parallel. Thus he proves the existence of parallel lines, not immediately but mediately, through one of the definition's formal effects.[23]

But St. Thomas, in the same place, generates another paradox when he says:

> The question of essence follows on the question of existence.[24]

This notion seems to suggest that Euclid should have proved the existence of parallels even before he stated what they are.

The solution of this difficulty lies in the distinction between *real* and *nominal* definition, as we shall see below. But a prior distinction is necessary, namely that between the *order of discovery* and the *order of exposition*.

3. Orders of Discovery and Exposition

Geometry, historically, does not begin with Euclid. Rather, Euclid placed the existing geometrical knowledge into scientific order, as Proclus says:

> . . . to make perfect the understanding of the learner in regard to the whole of geometry.[25]

In so doing, Euclid evaded the merely probable and sometimes fallacious[26] arguments advanced by his predecessors. Euclid's work has even been lauded as *the* model and inspiration for

[23] Cf. Heath, *Euclid*, Vol. I, p. 316; Heath points out that Euclid by proving Prop. 30 first, excludes the possibility of there being more than one parallel line through a given point.

[24] *Summa Theologiae*, I, Q. 2, art. 2, ad 2.

[25] Proclus, *Commentary*, p. 68.

[26] Cf. Heath, *Euclid*, Vol. 1, p. 202.

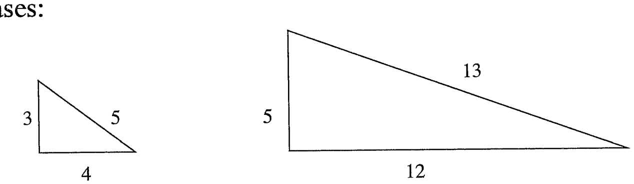
all subsequent attempts to systematize human knowledge, as one modern scientist notes:

> We reverence ancient Greece as the cradle of Western science. Here for the first time the world witnessed the miracle of a logical system which proceeded from step to step with such precision that every one of its deduced propositions was absolutely indisputable—I refer to Euclid's geometry. This admirable triumph of reasoning gave the human intellect the necessary confidence in itself for its subsequent achievement.[27]

Yet the order in which Euclid proves his propositions is not the order in which those propositions were discovered. Pythagoras, for example, who antedates Euclid by almost three hundred years, is credited with discovering the following theorem from the *Elements*:

> In right-angled triangles, the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.[28]

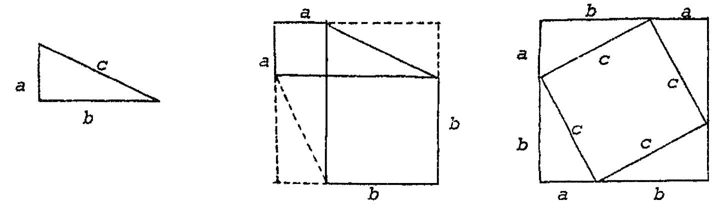
By assigning certain whole numbers as measures of the sides, one can easily see that the proposition is true in at least two cases:



9 + 16 = 25	25 + 144 = 169
(3x3) + (4x4) = (5x5)	(5x5) + (12x12) = (13x13)

These cases generate a *suspicion* that the proposition holds universally, but this requires further proof:

[27] Albert Einstein, "On the Method of Theoretical Physics," p. 13.
[28] *Elements*, Bk. I, Prop. 47.



Beginning with any right triangle with sides *a* and *b*, and hypotenuse *c*, construct squares on *a* and *b* and complete the square with side $(a + b)$. Construct another square with side $(a + b)$, this time containing the square on *c*, as shown. Thus the two squares are equal and each contain the original triangle four times. Subtracting these triangles from each square, the remainders are equal. Thus the square on *a* plus the square on *b* equals the square on *c*, the hypotenuse, which was to be proved.

This method of proof, although persuasive, lacks the rigor of scientific demonstration because it relies in large measure on the imagination. In contrast, Euclid's proof occurs as the forty-seventh proposition of the first book and depends on most, if not all, of the preceding forty-six propositions. Yet the method above, or one like it, was probably the first attempt at a theoretical proof of this particular problem.

It is likely that most of the propositions of geometry have undergone a similar development from mere suspicion or educated guess to formal proof. Thus we can discern a certain *order of discovery* by which the human mind advances toward scientific knowledge. This order is distinct from, and often, opposed to, the *order of exposition* by which one who already has science, Euclid for example, can impart his knowledge to another.[29]

[29] Cf. *Republic*, Bk. VI (510b) p. 745; Plato here makes a somewhat obscure reference to this distinction: "There is one section of it [the divided line] which the soul is compelled to investigate . . . by means of assumptions from which it proceeds not up to a first principle, but down to a conclusion, while there is another section in which it advances from its assumptions to a beginning or principle that transcends assumption,

This distinction is important for our investigation of definition. For we have said above that definition must be "through terms that are prior and more intelligible." Optimally, following the order of exposition, we will define by what is prior *absolutely*. But it is often necessary to follow the order of discovery and define by what is prior *with respect to us*. Aristotle provides an illustration:

> Absolutely, the prior is more intelligible than the posterior, a point, for instance, than a line, a line than a plane, and a plane than a solid. . . .
>
> Whereas to us it sometimes happens that the converse is the case: for the solid falls under perception most of all —more than a plane and a plane more than a line, and a line more than a point: for most people learn things like the former earlier than the latter.[30]

Thus definitions are sometimes given which define by what is scientifically posterior:

> Among definitions of this kind are those of a point, a line, and a plane, all of which explain the prior by the posterior; for they say that a point is the limit of a line, the line of a plane, the plane of a solid.[31]

For this reason, Euclid supplements the scientific definitions of point, line, surface and solid by a description more proportionate to a beginner in the science:

(i)	A *point* is that which has no part.	[definition]
(ii)	A *line* is breadthless length.	[definition]
(iii)	The extremities of a line are points.	[description]
(iv)	A *surface* is that which has length and breadth only.	[definition]
(v)	The extremities of a surface are lines.	[description]

and in which it makes no use of images . . . relying on ideas only and progressing systematically through ideas."

[30] *Topics*, Bk. VI, Chap. 4 (141b 6–13).

[31] *Loc cit.*, (141b 19–22).

(vi)	A *solid* is that which has length, breadth and depth.	[definition]
(vii)	An extremity of a solid is a surface.[32]	[description]

The distinction between the orders of discovery and exposition having been noted, let us turn to the distinction between real and nominal definition.

4. *Real and Nominal Definition*

According to Aristotle, definition properly speaking is "an indemonstrable positing of essential nature."[33] But in the same chapter, Aristotle refers to another kind of definition which is "a statement of the meaning of the name or of equivalent nominal formula [*logos heteros onomatodes*]".[34] The first is called *real*, the second, *nominal* definition. The need for the latter is due to one of the two defects in our knowledge: either ignorance of the existence of the thing defined, or ignorance of its essence.

If we do not yet know whether a thing exists, we cannot posit anything about what it is. There can be no discussion, for example, of whether unicorns can have more than one horn. Since no unicorns are known to exist, we can only refer the questioner to what is generally meant by the name, an animal which, among other things, has only one horn. This is the reason for the statement of St. Thomas quoted (in part) above:

> In order to prove the existence of anything, it is necessary to accept as a middle term the *meaning of the word*, and not its essence, for the question of its essence follows on the question of its existence.[35]

[32] *Elements*, Bk. I, Defs. 1, 2, 3, 5, 6; Bk. XI, Defs. 1, 2.

[33] *Posterior Analytics*, Bk. II, Chap. 10 (94a 11–2).

[34] *Loc cit.*, (93b 29–30).

[35] *Summa Theologiae*, I, Q. 2, art. 2, ad 2 (emphasis added).

Once the existence of something, say a unicorn, has been established, it becomes possible to settle questions about its attributes by reference to its essence rather than to the meaning of its name. Until the existence of the unicorn is shown, it is impossible to have anything more than a nominal definition. However, when something nominally defined can be proved to exist, as in geometry, the nominal definition may become real.

It should be noted that if the essential nature has been correctly assigned in the nominal definition, the transformation to a real definition is only formal, i.e., the terms of the definition are not altered. A material change is occasioned only when an initial ignorance of the thing's essence needs to be rectified, as we shall see.

It is possible to have some notion of a thing without knowing its essence perfectly, as St. Thomas argues:

> If no other notion could be had of a thing except the definition, it would be impossible for us to know that some thing is, without knowing the essence of it. . . . For in regard to a thing completely unknown to us, we cannot know if it is or not. But we do find some other notion of a thing besides the definition, namely a notion which explains the *signification of the name.*[36]

In cases of this kind, it is possible to use the notion by which the thing is known as a provisional definition and argue from it to its cause and from the cause to a proper definition. To use an example from the *De Anima*,[37] *anger* can be defined nominally as "the appetite for returning pain for pain" or "a boiling of the blood surrounding the heart." However, the *cause* of the boiling is "the appetite for returning pain for pain." Thus we have advanced toward a more proper definition of anger.

The distinction between real and nominal definition is of no small importance. Failing to make this distinction, some philosophers, both ancient and modern, have asserted with disastrous consequences that all definitions are nominal.

[36] *In II Post. Anal.*, Lect. 8, n. 484 (emphasis added).

[37] Cf. *De Anima*, Bk. I, Chap. 1 (403a 25–32).

Thomas Hobbes provides one modern example:

> And therefore in geometry . . . men begin at settling the *significations of their words*: which settling of significations they call *definitions*, and place them in the beginning of their reckoning.[38]

This position leads to a serious paradox, for which J. S. Mill takes Hobbes to task:

> It had been handed down from Aristotle, and probably from earlier times, as an obvious truth, that the science of geometry is deduced from definitions. This, so long as a definition was considered to be a proposition "unfolding the nature of the thing," did well enough. But Hobbes followed and rejected utterly the notion that a definition declares the nature of the thing, or does anything but state the meaning of a name; yet he continued to affirm as broadly as any of his predecessors that the ἀρχαί [*archai*], *principia*, or original premises of mathematics, and even of all science, are definitions; producing the singular paradox that systems of sci-

[38] *Leviathan*, Part I, Chap. 4; The absurdity of Hobbes' position when carried to its logical conclusion is accurately portrayed in Chapter VI of Lewis Carroll's *Through the Looking-Glass*, when Alice and Humpty Dumpty discuss the signification of 'glory':

"There's 'glory' for you."

"I don't know what you mean by 'glory'," said Alice.

Humpty Dumpty smiled contemptuously. "Of course you don't till I tell you. I meant 'there's a nice knock-down argument for you."

"But 'glory' doesn't mean 'a nice knock-down argument'," Alice objected.

"When *I* use a word," Humpty Dumpty said, in rather a scornful tone, "it means just what I choose it to mean—neither more nor less."

"The question is," said Alice, "whether you *can* make words mean so many different things.

"The question is," said Humpty Dumpty, "which is to be the master—that's all."

entific truth, nay, all truths whatever at which we arrive by reasoning, are deduced from the arbitrary conventions of mankind concerning the significations of words.[39]

It is interesting to note that Mill's argument closely resembles the last of three refutations which Aristotle advanced against Hobbes' ancient counterparts.[40] Both argue that an absurdity results from maintaining both of the following propositions:

(i) All definitions are nominal.
(ii) The first principles of every science are definitions.

Aristotle rejects the first proposition. Mill, on the other hand, rejects the second, thereby perpetuating and magnifying the original error of Hobbes.[41]

Let this suffice for an examination of the methods of defining common to the sciences. Although we have taken examples from mathematics, no attention has been given to the methods peculiar to mathematics itself. Therefore, the locus and the use of motion in mathematical definition will be considered in the following chapter.

IV. Definition in Geometry

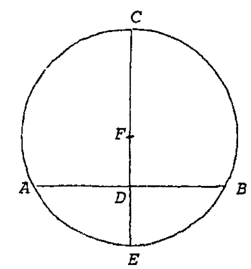
1. *Definition by the Locus*

Because the nature of the continuous involves the infinite, there is always something intriguing and perplexing in coming to grips with it. A sign of this is the fascination which thinkers of every age have displayed for Zeno's paradoxes.

[39] J. S. Mill, *System of Logic*, Bk. I, Chap. 8 (quoted in Heath, *Euclid*, Vol. I, p. 145).
[40] Cf. *Posterior Analytics*, Bk. II, Chap. 7 (92b 32); see also *In II Post. Anal.*, Lect. 6, n. 468.
[41] Cf. Mortimer J. Adler, "Little Errors in the Beginning," Prof. Adler here shows how errors of this kind run rampant through modern philosophy.

For this reason, one of the most interesting concepts in geometry is that of the locus, for here the infinity of magnitude clearly manifests itself, as will be evident in the examples which follow. Before examining the use of the locus in definition, we should first investigate the locus itself.

First, we will consider an important proposition from Euclid:



"To find the center of a given circle:[1]

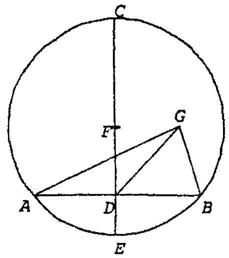
Let ABC be the given circle; thus it is required to find the center of the circle ABC.

Let a straight line AB be drawn through it at random, and let it be bisected at D; from D let DC be drawn at right angles to AB and let it be drawn through to E; let CE be bisected at F; I say that F is the center of the circle ABC.

For suppose it is not. . ."

Evidently, what Euclid intends here is to prove that F is the center of the circle by *reductio ad absurdum*.[2] It seems as though he would have to go through the remaining points, one by one, to show that they cannot be the center. But since there is a potentially infinite number of points in the circle besides F, such a task would obviously be impossible. Observe how Euclid in fact concludes the proof:

[1] *Elements*, Bk. III, Prop. 1.
[2] He can immediately exclude all the other points on CE, since, if the center is anywhere on the line, it will be at the point of bisection, F.



I say that F is the center of the circle ABC.

For suppose that it is not, but, if possible, let G be the center, and let GA, GD, GB be joined.

Then since AD is equal to DB, and DG is common, the two sides AD, DG are equal to the two sides BD, DG respectively; and the base GA is equal to the base GB, for they are radii; therefore the angle ADG is equal to the angle GDB. (I. 8)

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right; (I. Def. 10)

therefore the angle GDB is right.

But the angle FDB is also right; therefore the angle FDB is equal to the angle GDB, the greater to the less: which is impossible.

Therefore G is not the center of the circle ABC.

Similarly we can prove that neither is any other point except F.

Therefore the point F is the center of the circle ABC. Q.E.F.

What is marvelous about this proposition is that Euclid, at a stroke, has not only excluded one point G, but *all such points* G, from being the center. The validity of the proof depends upon the feature common to all the possible points G, namely, that the line drawn from G to D makes an oblique angle with the line AB.

Whenever all the points in a given area or on a given line share such a common feature, the area or line is called the *locus* (Latin for 'place') of all such points. In the example above, the

locus of all points G is the area within the circle, excluding the line CE.

Starting with a locus, any number of its points may be found. But it is impossible to construct[3] a locus from its points, no matter how many of them we begin with. The locus must first be constructed by means independent of its points. Then it must be shown that it contains all the desired points and only such points. The method of constructing a locus will be considered in subsequent chapters.

The simplest example of a locus used in a definition is that of the circle itself:

> A *circle* is a plane figure contained by one line [circumference] such that all the straight lines [radii] falling upon it from one point [center] among those lying within the figure are equal to one another.[4]

It is noteworthy, first of all, that this is not a proper but a nominal definition. It defines the circle by its parts and their relation, which relation is clearly posterior to the whole. In other words, we are not given its essence, but an essential property through which the geometer can demonstrate other properties.

Secondly, it is actually the *circumference*,[5] not the *circle*, which is being defined by a locus, even though this is not explicitly stated.

Since the radii are unlimited in number, their end points are unlimited in number. But all such points, by definition, must lie on the circumference. The circumference, then, is the locus of these points.

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[3] I mean *construct* in the classical Euclidean sense. In modern analytical geometry, an equation would suffice to produce, by fiat, any desired area. The expression $x^2 + y^2 \leq r^2$, for example, "produces" a circle.

[4] *Elements*, Bk. I, Def. 15.

[5] Modern usage of the term "circle" often, though not always, corresponds to Euclid's definition of the circumference, and even Euclid uses the word in that sense. Cf. Bk. III, Prop. 10; see also Smith, *History of Mathematics*, Vol. II, p. 278, *s.v.* "Circle."

Compare Euclid's definition with a more current one which, although inferior from the point of view of elementary geometry, does bring out more explicitly the notion of locus:

> [A circle is] a closed plane curve every point of which is equidistant from a fixed point within the curve.[6]

Curiously, neither definition invokes shape, that is, the fourth 'species' of quality. The boundary or termination of quantity is denoted here rather by relation or position. Proclus even defines a locus as "a *position* of a line or surface producing one and the same property."[7] Of course, in order for the line to have this property, it must also have a certain shape. But what is important here is that shape is not included in the definition. Hence, the locus shows what was pointed out before, namely, that quality is not considered in geometry *as such*, but only insofar as it designates the termination of quantity.

This introduction should suffice to make the basic notion of the locus clear. The more difficult cases will be taken up in the following sections.

2. *"Geometrical" and "Mechanical" Curves*

At the beginning of his treatment of curved lines in the second book of *La Géométrie*, Descartes brings up the distinction between "geometrical" and "mechanical" curves, to which we ought to give some consideration. The "mechanical" curves introduce the controversy over the admissibility of motion into geometry.

Descartes first points out that the ancients divided geometrical constructions into three kinds, *plane, solid*, and *linear*, according to the kinds of curves by means of which the desired construction is accomplished. In the first case, only straight

[6] *Webster's Third International Dictionary*, Vol. II, p. 408, *s.v.* "Circle."

[7] Proclus, *Commentary*, pp. 394–95 quoted in Heath, *Euclid*, Vol. I, p. 329 (emphasis added); cf. *Categories* (5a 15–29).

lines and circles are required; in the second, conic sections must be used as well.[8] Thus, the inscription of the cube in a sphere is a *plane* construction, even though the figure constructed is a solid. On the other hand, the duplication of the cube[9] is a *solid* construction since it requires the intersection of two parabolas in addition to the use of straight lines and circles. The origin of these technical terms is not altogether clear.

Concerning the third kind of construction, Descartes here remarks only that they require lines which are "more complex." Pappus, however, in treating the same threefold division, provides some interesting observations:

> There remains a third class which is called *linear* because other "lines" than those I have just described [straight lines, circles, conic sections], having diverse and more involved origins, are required for their construction. Such lines are the *spiral*, the *quadratix*, the *conchoid* and the *cissoid*, all of which have many important properties.[10]

In order to understand this third category of construction, it will be necessary to see in what way each of these lines is "more complex" or "more involved."

The Spiral (Archimedes, third century, B.C.)

Although the Archimedean *spiral*, or *helix*, is not the only kind of spiral, it is the most simple. We will take the definition given by Archimedes himself:

> If a straight line drawn in a plane revolve at a uniform rate about one extremity which remains fixed and return to the position from which it started, and if, at the same time as the line revolves, a point move at a uniform rate along the straight line beginning from the extremity which remains

[8] These terms may cause some confusion.

[9] Cf. Smith, *History of Mathematics*, Vol. ii, p. 313.

[10] *Pappi Alexandrini Mathematicae Collectiones*, Vol. I, Bk. III, Prop. 5 (quoted in Smith and Latham, *Le Géométrie*, p. 40).

fixed, the point will describe a *spiral* in the plane.[11] (see Fig. 1)

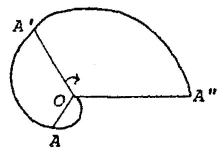

Fig. 1 The Spiral

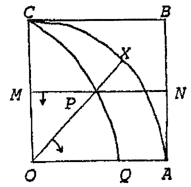

Fig. 2 The Quadratix

The Quadratix (Hippias, fifth century, B.C.)

The quadratix was developed in order to square the circle. The classic problem, of course, was to do so using straight lines and circles only. This curve is described as follows:

> In this figure, X is any point on the quadrant AC. As the radius OX revolves at a uniform rate from the position OC to the position OA, the line MN moves at a uniform rate from position CB to the position OA, always remaining parallel to OA. Then the locus of P, the intersection of OX and MN, is a curve CQ (the quadratix).[12] (see Fig. 2)

The Conchoid (Nichomedes, second century, B.C.)

The conchoid solves the problem of trisecting the angle, again without the classical restrictions. It is described as follows:

> We take a fixed point O which is d distant from a fixed line AB, and we draw OX parallel to AB and OY perpendicular to OX. We then take any line OA through O, and on OA produced, lay off AP=AP′=<u>k</u>, a constant. Then the locus of points P and P′ is a conchoid.[13] (see Fig. 3)

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[11] *On Spirals*, Def. I.

[12] *History of Mathematics*, Vol, II., p. 300.

[13] *Loc cit.*, Vol., II, pp. 298–99.

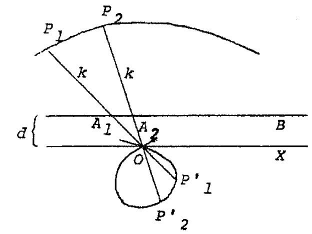


Fig. 3 The Conchoid

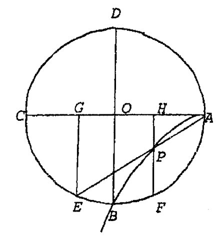


Fig. 4 The Cissoid

The Cissoid (Diocles, second century, B.C.)

The cissoid was developed to find the two geometric means between two given magnitudes, or, if one of these magnitudes is the Cartesian "unit," to find the cubic root of the other magnitude. The cubic root, in turn, is used in the classic problem of duplicating the cube. If the side of the cube to be duplicated is the unit, then a cube constructed with the cubic root of 2 as a side will have a volume double the original cube.

The cissoid is described as follows:

> Let AC, BD be diameters at right angles in the circle with center O. Let E, F be points on the quadrants BC, BA respectively, such that the arcs BE, BF are equal. Draw EG, FH perpendicular to CA. Join AE, and let P be its intersection with FH.
>
> The cissoid is the locus of all the points P corresponding to different positions of E on the quadrant BC and of F at an equal distance from B along the arc BA.[14] (see Fig. 4)

Motion is explicitly mentioned only in the definitions of the first two curves. Nevertheless, the *generation* of all four would seem to require at least one motion. In the case of the spiral and quadratix, two independent but simultaneous uniform motions are required for the generation of the curves. In the cissoid, two independent motions are required, but providing they are simultaneous, they need not be uniform.

---

[14] Heath, *Euclid*, Vol. I, p. 164.

Finally, the conchoid requires only one motion, that of the line AP, and thus uniformity is irrelevant.

One is therefore led to believe that the constructions involving such complex lines constitute a distinct type because these lines, unlike the circle or the conic sections, depend on motion for their generation. They would therefore belong to geometry only derivatively, being more properly considered in one of the *scientiae mediae*. If this is the case, it is clear why the ancients would have called the complex curves "mechanical."

Yet Descartes claims not to know why the ancients distinguished the complex curves in this way:

> And I do not understand why the [ancients] named them [the complex curves] mechanical rather than geometrical.[15]

Nevertheless, he advances five possible interpretations of "mechanical," none of which, in his opinion, are sufficient grounds for excluding all the complex curves from geometry. Curiously, the interpretation suggested above is not one of those considered.[16] In stating the first three interpretations, he argues that none of them is what the ancients had in mind.

The first interpretation he gives is apparently based upon the root of the word "mechanical":

[15] *La Géométrie*, p. 315 (my translation).

[16] One might conjecture that the idea of a *scientia media* in St. Thomas' sense of the term had been lost in the decadence of fifteenth and sixteenth century scholasticism, and hence, would not have occurred to Descartes. But the notion of mechanics as the application of pure geometry to physical objects survived in learned circles well past the publication of *La Géométrie* in 1637. Robert Boyle wrote in 1671: "I do not here take the term, *Mechanicks*, in that stricter and more proper sense, wherein it is wont to be taken, when 'tis used only to Signify the Doctrine about the Moving Powers (as the Beam, the Leaver, the Screws, and the Wedg) . . . but . . . in a larger sense, for those Disciplines that consist of the Applications of pure Mathematics to produce or modifie motion in inferior bodies.

> To say that it was on account of the need to use some machine to describe them, one must, for the same reason, exclude circles and right lines; seeing that one does not describe the latter on paper save with a compass and rule, which could be called machines as well.[17]

The second interpretation is based upon the various degrees of exactness required in the two disciplines:

> Neither is it because the instruments used to draw them, being more complicated than the rule and compass, cannot be as exact; since for this reason one should exclude them from mechanics, which demands exactness in that which is drawn by the hand, rather than from Geometry, where one seeks only exactness in reasoning. For the latter can undoubtedly be had as completely with these (complex) lines as with the others.[18]

It seems that Descartes neglects the fact that in mechanics, as in other sciences, there is both an order of discovery and an order of exposition. According to the first, we must have, as Descartes suggests, "exactness in what is drawn by hand." For example, when investigating the laws of motion governing the pendulum, one must be meticulous in graphically representing the results of measurements in order to see that the pendulum obeys a sine function. But once this discovery has been made, one can then apply, following the order of exposition, all the rigors of mathematical reasoning to the physical motion. It is only the latter which is the science of mechanics, properly so-called. Thus, the exactness of the hand is incidental to mechanics, and cannot be used to distinguish that science from geometry.

The third interpretation is based upon the desirability in geometry of requiring as few postulates as possible:

[17] *La Géométrie*, p. 315 (my translation).

[18] *Ibid.*, pp. 315–16 (my translation).

Nor would I say the cause to be that they did not want to multiply the numbers of postulates, and that they were content that one concede the possibility of joining two given points with a right line and describing a circle with a given center and passing it through a given point. For they had no qualms whatsoever about supposing [a postulate] other than these in order to treat the conic sections: that any given cone can be cut by a given plane.[19]

After giving his own view of the proper distinction between geometrical and mechanical curves, to which we will return, he gives two reasons why the ancients may actually have excluded linear constructions from geometry.

The first is based upon the way in which motion is involved in the consideration of these lines:

But perhaps what prevented the ancient Geometers from admitting those [lines] more complex than the conic sections is that the first of these considered by them were, as chance would have it, the Spiral, the Quadratix, and the like.

Now these in fact appear to belong only to Mechanics and are not in the least to be numbered among those that I think ought here to be admitted. For one imagines them described by two separate motions having no relation between themselves which one can measure exactly. But having afterward examined the Conchoid and Cissoid and some few others which are [constructed] of them, they nonetheless made no more account of these than they did of the first, perhaps because they did not sufficiently discern their properties.[20]

The second reason supposes that the ancients acted upon a prudential decision to follow the proper order of investigation:

Or else it is that, seeing themselves to know but a few things concerning the conic sections and that much still remained unknown concerning what can be made with the rule and compass, they believed that more difficult matters ought not to be engaged in.[21]

[19] *Ibid.*, p. 316 (my translation).

[20] *Ibid.*, pp. 316–17 (my translation).

It is the first of these two alleged reasons which is most illuminating here. It is apparent upon careful examination of the four complex curves described above that the spiral and the quadratix depend upon motion far more than the conchoid or the cissoid. Although, as noted above, all four must be *constructed* with motion, the latter need not be *defined* with motion, but may be defined as loci of points. On the other hand, the inclusion of motion in the definitions of the spiral and quadratix is essential. A sign of this is the necessity of positing a uniform velocity in both the definition and the construction of the spiral and quadratix, while in the construction of the conchoid and cissoid the velocity is incidental. The ancients, claims Descartes, did not appreciate this distinction and hence, were reluctant to admit any of these curves into pure geometry.

Descartes' assumption is that there are some conditions under which motion may be admitted into pure geometry. This contention is supported by the inclusion into geometry of the so-called "solids of revolution" by Euclid and Apollonius. The status of these solids, therefore, ought to be investigated next in order to determine the validity of Descartes' claim.

3. *The Solids of Revolution*

Euclid is sometimes criticized by modern writers for his use of motion in defining the so-called solids of revolution, i.e., the sphere, the cone and the cylinder. Such a reference to motion, they claim, constitutes a "lapse" in his otherwise pure geometry:

One even notices such a lapse in Euclid wherein he makes use of ideas of motion and the translation of bodies.[22]

[21] *Ibid.*, p. 317 (my translation).

[22] John F. Kiley, *Einstein and Aquinas: A Rapprochement.*

These definitions should therefore be examined to see whether, and to what extent, Euclid is guilty in this regard. The definitions will first be presented, then commented upon, special attention being given to that of the sphere.

The sphere is the first solid of revolution which Euclid defines:

> When, the diameter of a semicircle remaining fixed, the semicircle is carried round and restored again to the same position from which it began to moved, the figure so comprehended is a *sphere.*[23]

The next such solid to be defined is the cone:

> When, one side of those about the right angle in a right-angled triangle remaining fixed, the triangle is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a *cone.*[24]

Finally, Euclid defines the cylinder:

> When, one side of those about the right angle in a rectangular parallelogram remaining fixed, the parallelogram is carried around and restored again to the same position from which it began to be moved, the figure so comprehended is a *cylinder.*[25]

It is clear that some motion is involved in each of these definitions, namely the revolution of the semicircle, of the right triangle and of the rectangle, respectively. But what is equally striking is the curious way in which the definitions are phrased. Neglecting his familiar pattern of genus and difference, Euclid has cast these in the form of a proposition, or more specifically, a construction.

This leads to an obstacle in our investigation. With the complex curves, we distinguished between motion in their very definition and motion only in their construction. But in

[23] *Elements*, Bk. XI, Def. 14.
[24] *Op. cit.*, Def. 18 .
[25] *Op. cit.*, Def. 21.

the case of the solids of revolution, this distinction is not so clear. The first question, it seems, is whether these are true definitions at all. The definition of the sphere will be considered first.

The sphere appears to be to the solids, analogously, what the circle is to the plane figures. One would expect, therefore, that the definition of the sphere be analogous to that of the circle [*see* p. 79]. This method, suggested by Aristotle in his *De Caelo*,[26] is given explicitly by Heron of Alexandria:

> A sphere is a solid figure bounded by one surface such that all the straight lines falling on it from one point of those which lie within the figure are equal to one another.[27]

This, of course, presupposes that Euclid's method of defining the circle is preferable to his method of defining the sphere. If, on the contrary, the latter method is preferable, one would expect an analogous definition of the circle:

> A circle is the figure described when a straight line, always remaining in one plane, moves about one extremity as a fixed point until it returns to its first position.[28]

Eliminating the anomaly in this way, however, merely extends to the circle the difficulty we originally encountered with the sphere, namely, that it does not appear to be a definition at all. Thus it appears that Heron's definition of the sphere, modeled on that of the circle, is more properly a definition than Euclid's.

Another anomaly is that the sphere, unlike most of those things defined in Euclid's *Elements*, is accompanied neither by a construction distinct from the definition, nor a postulate asserting the sphere's existence. Nor is the definition itself, strictly speaking, a construction, since no attempt is made to prove that what has been constructed is in fact a sphere.

[26] *De Caelo*, Bk. II, Chap. 14 (297a 24).
[27] Quoted in Heath, *Euclid*, Vol. III, p. 269.
[28] Heath, *Euclid*, Vol. I, p. 184.

Even so, such a proof would seem to require a prior definition of the sphere and a postulate which allows the revolution of any given plane figure about one of its sides.

If the definition of the sphere is neither a construction nor a definition proper, the possibility remains that the sphere is a *primary object* whose existence and essence are both assumed. Since the circle, as noted above, is a primary object in plane geometry, it would not be surprising to discover something similar about the sphere. But although the essence and the existence of the circle are both assumed, they are not assumed at the same time. First the circle is defined, then its existence is postulated, since there is nothing prior out of which it can be constructed. Thus, it seems, this last possibility must be excluded as well.

However, two of the things we have just said concerning the circle should be well noted: first, it is a primary object in *plane geometry*; and second, it is prior to every construction. Reflection on these two points will assist in resolving our difficulties with the sphere.

The science of plane geometry has for its *genus subjectum* magnitude extended in two dimensions: length and breadth. Solid geometry, on the other hand, considers magnitude extended in three dimensions: length, depth and breadth. Now, two-dimensional objects are abstracted from three-dimensional objects (*abstractio a materialibus condicionibus*), while the latter are abstracted from sensible things (*abstractio a materia sensibilis*).[29] Thus, in the order of discovery, two-dimensional objects are posterior to, and more abstract than, three-dimensional objects. Because their objects differ in degree (and kind) of abstraction, the sciences of plane geometry and solid geometry can therefore be distinguished:

> Speculative sciences are differentiated according to their *degree of separation from matter* and motion.[30]

[29] For the various meanings of *abstractio*, see Deferrari, pp. 12–13.
[30] *In Boethium de Trinitate*, Q. 5, art 1., corp. (emphasis added).

Two-dimensional figures are nevertheless prior in the order of exposition. Thus plane geometry precedes solid geometry in the proper order of acquiring the sciences. But lest the solid geometer be required to prove the propositions of plane geometry all over again, he assumes the conclusions of that science as principles in his own. Thus solid geometry is a derivative science with respect to plane geometry. And Plato was rightfully at pains to distinguish the two in the *Republic*.[31]

Returning to our second observation on the circle, it is now apparent that the analogy between the circle and the sphere is not as complete as was originally assumed. Whereas the circle is a primary object in a primitive science, the sphere is only primary in a derivative one. The circle cannot be understood or constructed by means of anything more simple or fundamental, either in plane geometry or in any other science. The sphere is the most simple and fundamental among solids, but it can be understood in terms of the circle, or more precisely, the semicircle. Similarly with the cone and cylinder.

Since the definitions of the sphere, cone and cylinder occupy a peculiar place in the science, it should not be surprising that the definitions are themselves peculiar. For they must be defined in such a way that the knowledge of plane geometry can be applied readily to solid figures by using the definitions as middle terms of demonstrations. This is accomplished by the inclusion of the semicircle, the triangle and the rectangle in the definitions of the sphere, cone and cylinder. These are nominal definitions appropriate for a beginner in the science of solid geometry. The most proper definition of the sphere remains the one given by Heron of Alexandria.

The significance of motion in the nominal definition is not the motion itself, but the unbroken continuum which the motions describe. Thus, once the solid is described, the disparity between the realms of plane and solid geometry is overcome, and the motion becomes irrelevant.

[31] *Republic*, Bk. VII (528b).

Euclid's definitions of the solids of revolution are called *genetic*, since they make things known by their genesis or origin. The consideration of origins is a dialectical tool common to the sciences, as is clear from Aristotle's *Politics*:

> He who considers things in their first growth and origin (γενεσις) [*genesis*], whether a state, or anything else, will obtain the clearest view of them.[32]

Thus Euclid's "lapses" are in fact only nominal definitions used as dialectical tools to bridge the gap between the conclusions of a primitive science and the principles of a derivative one.

Let us return, therefore, to the complex curves, to see whether their use of motion can be justified similarly.

4. *The Complex Curves*

The sphere, cone and cylinder, as we have seen, need not be defined with motion. In this respect, they resemble the conchoid and cissoid rather than the spiral and quadratix. And even though the solids can be generated with motion, as are the complex curves, they need not be generated at all, since they are the primary objects of their science.

The sphere, cone and cylinder are simple figures which present themselves naturally and immediately to even the most youthful and untutored mind[33] as objects of contemplation. Their properties, however, can be discovered only by patient investigation. The spiral, quadratix, conchoid and cissoid, on the other hand, appear to be artificial constructs whose properties are useful for particular purposes: squaring the circle, trisecting the angle, etc. They are understood only by those advanced in the geometrical sciences. They are neither primary objects nor are they constructible from those which are.

[32] *Politics*, Bk. I, Chap. 1 (1252a 24–25).

[33] Cf. *Ethics*, Bk. I, Chap. 3 (1095a 2–5).

The artificiality of the complex curves renders their place in the mathematical sciences dubious. Galileo excludes them from his mathematical considerations of motion, in spite of the fact that mathematicians "have very commendably established the properties which these curves possess in virtue of their definitions,"

> And first of all it seems desirable to find and explain a definition best fitting *natural* phenomena. For anyone may invent an arbitrary type of motion and discuss its properties; thus, for instance, some have imagined helices [spirals] and conchoids as described by certain motions *which are not met with in nature.*[34]

It is plausible that the ancients shared Galileo's apprehension about the complex curves when they relegated to the category of "linear" any construction employing them. Descartes' distinction between the spiral and the quadratix on the one hand, and the conchoid and cissoid on the other, is an important one. Yet it serves to *distinguish* the complex curves rather than to justify their inclusion into geometry proper.

Our investigation of the complex curves brings to light a point which is of great importance in understanding the principles of mathematics. The mathematical sciences, like all theoretical sciences, examine an order which man neither brings about nor changes. This distinguishes the theoretical sciences from the practical sciences and the arts. Even though geometry employs constructions, they are ordered to the knowledge acquired as a result.

The introduction of complex curves into geometry as objects just as worthy of contemplation as the circle, triangle, sphere and cone, undermines the basis in nature, and hence, the certainty of the science. Geometry would become, as

[34] *Two New Sciences*, Third Day, n. 197, p. 153 (emphasis added).

Hobbes and Mill have suggested, a series of purely logical deductions from arbitrary premises. Although this view of geometry does not appear explicitly in Descartes' *La Géométrie*, it seems to be characteristic of post-Cartesian mathematics.[35]

V. Definitions of the "Conic Sections"

1. *Apollonius: Sections of a Cone*

The curves known as "conic sections," the hyperbola, the parabola, and the ellipse, take their generic name from one of the earlier methods of defining them, namely, the intersection of a conic surface with a cutting plane.

Undoubtedly, the best-known formulations of the definition by conic section are those of Apollonius given in his treatise *On Conic Sections*. Apollonius "defines" the parabola, hyperbola and ellipse as constructions, much the same way that Euclid "defines" the solids of revolution. While there is no motion involved *per se* in conic sections, the operation of "cutting" a cone by a plane is simply asserted. As we shall see, this operation is necessary in order that the conclusions of elementary geometry can readily be applied to the investigation of these curves.

The definitions of the hyperbola, parabola and ellipse all suppose a cone "cut by a plane through its axis," that is, through the line drawn from the vertex to the center of the circle which forms its base. This plane does not produce the curves, but rather an *axial triangle* which becomes extremely useful for defining, and later, investigating the conic sections.

[35] E.g., mathematics viewed as an axiomatic system by Kant.

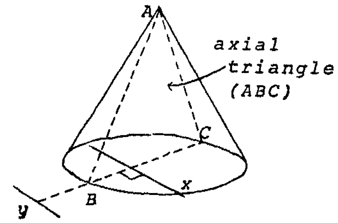


Fig. 1

Apollonius imagines the second cutting plane to intersect the plane of the cone's base in a line perpendicular to the base of the axial triangle. (see Fig. 1) If the line of intersection falls beyond the base of the cone [y], the section will be an ellipse. If it falls within the base [x], it will be either a parabola or an hyperbola. Since he distinguishes the latter two by reference to their "diameters," it will be helpful to determine what he means by the term:

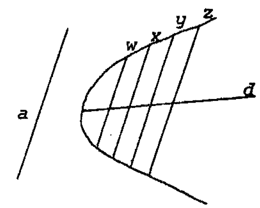


Fig. 2

> Of any curved line which is in one plane, I call that straight line the *diameter* [d in Figure 2] which, drawn from the curved line, bisects all straight lines [e.g. w, x, y, z] drawn to this curved line parallel to some straight line [a].[1]

If then, the diameter of the curve produced by the surface of the cone and the second cutting plane is parallel to one of the sides of the axial triangle, it is a parabola. (see Fig. 3) If when produced, however, it meets one of those sides, it is a hyperbola. (see Fig. 4)

[1] *On Conic Sections*, Bk. I, Definition 4.

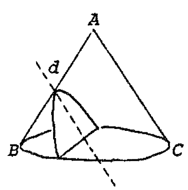


Fig. 3

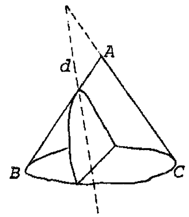


Fig. 4

With these things in mind, let us examine the precise formulations which Apollonius himself lays out:

> Parabola: If a cone is cut by a plane through its axis, and also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if further the diameter of the section is parallel to one side of the axial triangle . . . such a section [is] called a parabola.[2]
>
> Hyperbola: If a cone is cut by a plane through its axis, and also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle beyond the vertex of the cone . . . such a section [is] called an hyperbola.[3]
>
> Ellipse: If a cone is cut by a plane through its axis and is also cut by another plane on the one hand meeting both sides of the axial triangle, and on the other extended neither parallel to the base nor subcontrariwise [in which two cases it will be a circle], and if the plane the base of the cone is in, and the cutting plane meet in a straight line perpendicular either to the base of the axial triangle or to it produced . . . such a section [is] called an ellipse.[4]

It should thus be clear that, for Apollonius, the intelligibility of the hyperbola, parabola and ellipse arise out of the prior intelligibility of the cone, at least as far as his treatise is

[2] *Loc cit.*, Prop. 11.
[3] *Loc cit.*, Prop. 12.
[4] *Loc cit.*, Prop. 13.

concerned. Using the cone, its axial triangle, and the definitions cited above, he is able to deduce a number of curious properties of the conic sections.

Let us now return to the geometry of Descartes.

2. Decartes: Loci of Points

Pappus, a Greek mathematician of the third century A.D., proposed a problem which subsequent geometers had not been able to solve using the methods of Euclid and Apollonius:

> If [four] straight lines are given in position, and if straight lines be drawn from one and the same point, making given angles with the [four] given lines . . . and if the rectangle of two of the lines so drawn bears a given ratio to the rectangle of the other two; then . . . the point lies on a solid locus given in position, namely, one of the three conic sections.[5]

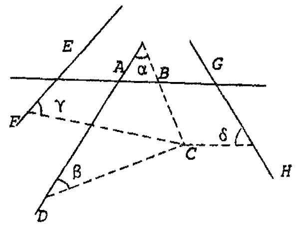
In other words, if lines AB, AD, EF and GH in the diagram at left are given in position,[6] angles α, β, γ, and δ are constant, and the ratio

rect. CB,CF : rect. CD,CH

is kept constant, the locus of all the points C will be one of the conic sections.

[5] *La Géométrie*, p. 306 [p. 21].

[6] To the extent that this statement of the problem anticipates the modern "Cartesian coordinate system," credit should be given to Pappus rather than Descartes. Even so, this coordinate system differs from the modern conception in three important ways: (1) there are four axes, not two; (2) the axes need not be perpendicular; and (3) there is no "fourth quadrant," i.e., no area defined by multiplying one negative quantity by another negative quantity, Descartes' English translators lamented that he was "not able to free himself from the old [Euclidean] traditions . . ." (Latham and Smith, p. 17, n. 26). However, they offer no proof that such quantities exist nor provide either a definition of them or a method of constructing them.



Descartes begins his solution to this problem by assigning to the unknown quantities AB and CB the values x and y. Then, assigning other letters [b, c, d, e, f, g and z] to represent the numerous known quantities given in the problem, he arrives at four equations for the four lines drawn from point G:

$$CB = y \qquad CF = \frac{ezy+dek+dex}{z^2}$$

$$CD = \frac{czy+bcx}{z^2} \qquad CH = \frac{gzy+fgl+fgx}{z^2}$$

By the terms of the problem, we know that CB·CF = CD·CH.[7] Thus, by substitution and simplification, we arrive at the following rather complicated equation:[8]

$$y^2 = \frac{(cfglz-dekz^2)y-(dez^2+cfgz-bcgz)xy+bcfglx-bcfgx^2}{ez^3-cgz^2}$$

Fortunately, by substituting 2m for $\frac{cflgz-dek^2}{ez^3-cgz^2}$ and $\frac{2n}{z}$ for this equation can be reduced to:[9]

$$y = m - \frac{n}{z}x + \sqrt{m^2 + ox + \frac{p}{m}x^2}$$

Descartes claims that it is "a simple matter" to determine the curves "in terms of Theorems 11, 12, and 13 of the first book of Apollonius."[10] How simple? "If the term $\frac{p}{m}x^2$ is zero, the

[7] It should be noted that Pappus stated the problem in terms of rectangles whereas Descartes restates it in terms of his own "multiplication of lines." It is beyond the scope of our present discussion to determine whether or not Descartes' restatement is ultimately "legitimate."

[8] *La Géométrie*, p. 325 (p. 60).

[9] *Loc cit.*, p. 326 (p. 63).

[10] *Loc cit.*, p. 332 (my translation).

[curve] is a parabola; if it is designated by a plus sign, it is an hyperbola; and finally, if it is designated by a minus sign, it is an ellipse."[11]

What is of interest here is how Descartes proves that one of three "properties" (the three possible versions of the equation) must belong to the locus described in the problem, and then refers to Apollonius to show that the hyperbola, parabola and ellipse are the only curves to which these properties belong.[12] (In certain special cases, the solution of the equation is a circle, which for our purposes can be regarded as merely a special case of the ellipse.)

While Descartes does not formally *define* the hyperbola, parabola and ellipse as loci of points, there is no reason in principle why one could not, since, as we have seen, the locus is a legitimate method of definition.

The locus in fact has become the standard method of definition in many textbooks of mathematics. The following is from *A First Year of College Mathematics*, published in 1937:

> A *conic section* (or *conic*) is the locus of [points] whose distance from a certain fixed point is in a constant ratio to its distance from a fixed straight line.[13]

One problem with such a definition is that the perfectly sensible notion of a "locus of points" too easily degenerates into the self-contradictory expression "an infinite set of points."

[11] *Loc cit.*, p. 328 (my translation).

[12] Despite its bewildering complexity, the argument of *La Géométrie* can be reduced to two first-figure syllogisms. Let A be "the solution of the four-line locus problem;" B, "the curves defined by Descartes;" C, "the curves defined by Apollonius;" D. "the conic sections."

Then:	(1)	A is B	(2)	A is C
		B is C		C is D
		A is C		A is D

The second conclusion is that thus the solution to the four-line locus problem is one of the conic sections.

[13] Brink, *A First Year of College Mathematics*, p. 399.

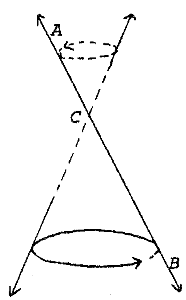
Such indefensible statements can be found in any number of other noteworthy textbooks.[14]

Yet a far more serious objection is that the natures of the hyperbola, parabola and ellipse are not illuminated by describing them as four-line loci or by reducing them to an equation. Rather, all three are thereby reduced to the status of artificial curves as intrinsically uninteresting as the conchoid, the cissoid or the quadratix. By contrast, when defined as sections of a cone, they seem to "come to life" as worthy objects of mathematical wonder. More will be said on this topic in the Epilogue.

Let us turn to the method of defining used by Pascal.

3. Pascal: Projection of the Circle

Pascal's *Generation of the Conic Sections* is the first draft of an uncompleted treatise. It is preserved only in a copy made by Leibnitz. He begins with a generation of the cone which differs from both Euclid's and Apollonius' in that the cone is "infinite" in extension.



> If a straight line, infinite in both directions [AB in the figure at left], is drawn from a point [C] taken outside of the plane of a circle to a point taken on its circumference, and if it is rotated about the circumference while the first point

[14] E.g., Hardy, *Pure Mathematics*, p. 24.

> remains fixed, the surface which this straight line describes in its rotation is called a conic surface. . . The line so taken and kept fixed in some arbitrary position of its rotation will be called the generatrix.[15]

With this in his imagination, Pascal further imagines himself as an observer situated at the vertex of the cone looking at the *projection* of the generating circle upon some plane beyond it. Of course, if the plane is parallel to the generating circle, the projection will be a circle as well. But if not, the projection, interestingly, will be an hyperbola, parabola or ellipse.

In this context, a projection is a species of locus. It is the locus of all points on the plane of projection which are colinear with both the vertex of the cone and any point on the generating circle.

The projection is an ellipse "if the plane of projection . . . is not parallel to any generatrix, that is to say, to any ray." It is apparent from Figure 2 that for every point on the circle there is a corresponding point on the ellipse.

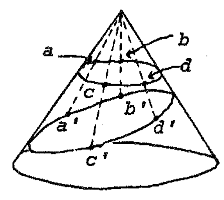


Fig. 2

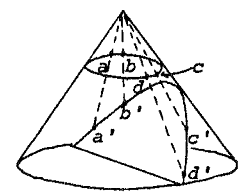


Fig. 3

Likewise, the projection is a parabola "if the plane of projection is parallel to one of the generatrices only, that is to say, to a single ray." Curiously, the point on the circle through which that one generatrix passes is the only one which has no corresponding point on the parabola. (see Fig. 3)

[15] This and all other citations from Pascal's treatise are from an anonymous unpublished translation in the possession of the author.

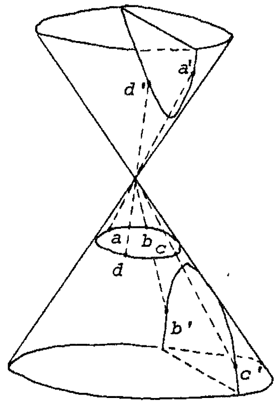


Fig. 4

Similarly, the projection is an hyperbola "if the plane of projection is parallel to two of the generatrices." There are three curiosities involved here. The first is that there are two "missing points," that is, two points on the circle which have no counterparts on the hyperbola. The second is that if the lines of projection are drawn down from the vertex through the circle, only one half of the hyperbola is produced and only one arc of the circle is projected. The other part of the hyperbola is obtained by drawing the lines of projection up from the remaining arc of the circle through the vertex into the upper half of the cone. The third is that the two "missing points" separate these two arcs of the projected circle.

What is interesting about Pascal's use of projection to define the conics is the way in which it unifies the three apparently heterogeneous curves. They can all be viewed as projections *of the same circle* upon three different planes. Thus, Pascal also avoids the arbitrariness of defining the curves by equations.

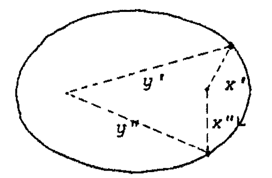
The major disadvantage of Pascal's approach is that there is nothing in his definitions which can be used as a middle term of demonstration. As a consequence, his treatment of the conics, unlike that of Apollonius, is not a progressive series of theorems, but a succession of scholia and corollaries. The

reader must rely upon his powers of imagination and intuition rather than discursive reasoning. In short, Pascal's definitions appear to be incapable of producing *scientific knowledge.*

It remains to consider the definition of conics by means of motion.

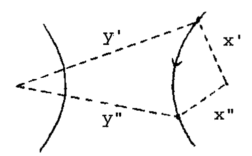
4. *Classical Mechanics: Paths of a "Moving Point"*

In his *Principia Mathematica*, Newton includes a number of propositions dealing with hyperbolic, parabolic and elliptical "orbits," that is, the paths of a "moving point" which constitute an hyperbola, parabola or ellipse.[16] While Newton does not explicitly *define* the conic sections at all, his method of *describing* them as "paths of a moving point" has had such an impact that a number of standard reference works have adopted it as a definition. The following definitions are taken from *Webster's Seventh New Collegiate Dictionary*:



ellipse: $y + x = k$

ellipse, a closed plane curve generated by a point moving in such a way that the sums of its distances from two fixed points is a constant.[17]

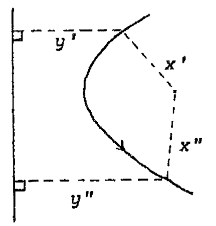


hyperbola: $y - x = k$

[16] See *Principia*, Bk. I, Section III, esp. Propositions 11, 12, 13.

[17] *Webster's Seventh New Collegiate Dictionary*, p. 268.

hyperbola, a plane curve generated by a point so moving that the difference of the distances from two fixed points is a constant.[18]



parabola: $y = x$

parabola, a plane curve generated by a point moving so that its distance from a fixed point is equal to its distance from a fixed line.[19]

This method of defining the conic sections is subject to the same critique given of the complex curves in Chapter IV. Like the conchoid and the cissoid, the hyperbola, parabola and ellipse can just as easily be defined as loci of points. But as long as the conic sections are defined with motion, they are mechanical in the sense of belonging more to the *scientiae mediae* than to pure geometry. Appropriately, the most interesting applications of the conic sections *qua* "paths of a moving point" are in Kepler's and Newton's work in the derivative science of astronomy and Galileo's study of projectile motion.[20]

5. *Epilogue*

In this chapter, we have laid out four ways in which the conic sections can be defined. It remains to determine the status of each method within the mathematical sciences, as well as the status of the conic sections themselves.

[18] *Op. cit.*, 408.

[19] *Op. cit.*, p. 610; the *Oxford English Dictionary*, on the other hand, defines the conic sections statically, i.e. as loci of points.

[20] *Two New Sciences*, The Fourth Day, *passim*.

As we have already noted, the use of motion places the definition of the conics as "paths of a moving point" outside of pure geometry, although the disciplines which employ this method fall within the *quadrivium* of the liberal arts.

Pascal's use of projection has the advantage of illustrating that the conic sections are neither completely arbitrary nor completely heterogeneous, but it lacks the advantage of providing a middle term of demonstration.

On the other hand, Descartes' equation derived from the terms of the four-line locus problem has the dual advantage of *a*) providing a middle term of demonstration; and *b*) showing the unity of the conic sections (all can be represented by the same equation). Yet there is nothing in the equation itself which suggests that the hyperbola, parabola and ellipse are any less arbitrary than the conchoid, cissoid or quadratix.

Only Apollonius' definition by conic section preserves in a unified way all three of the advantages found severally in Pascal and Descartes. It must be admitted, however, that the unity of the hyperbola, parabola and ellipse are more striking in Pascal than in Apollonius, and that Descartes' equation is easier to manipulate and wider in its application than Apollonius' cumbersome apparatus.

Turning now to the conic sections themselves, it should be observed that, while they are certainly made more intelligible by being defined as sections of a cone or projections of a circle, the fact that they can be defined independently as loci of points or paths of a moving point illustrates that they are *primary objects* of a science, namely, the branch of geometry called *conics*.

The conic sections cannot be constructed from anything more simple. In fact, just as the primary objects of plane geometry (lines and circles) are necessary and sufficient for all constructions which the ancients called *plane*, so likewise the primary objects of conics (hyperbolas, parabolas, and ellipses) are necessary and, along with lines and circles, sufficient for all constructions called *solid*.

What remains disconcerting is that the hyperbola, parabola and ellipse are not nearly as accessible to the imagination's grasp as other primary objects, like the sphere, cylinder and cone. In fact, it is precisely this inaccessibility which makes Descartes' equation so attractive. But because the conic sections *are* primary objects, they possess a virtually inexhaustible number of interesting properties yet to be discovered and thus remain a standing challenge to man's sense of wonder.

It has been argued in this article that there is a distinction between *arbitrary* curves like the conchoid, cissoid and quadratix which men have fabricated for specific purposes, and *non-arbitrary* curves which have been *discovered*, whose properties can be explored, and which belong to an order of things which men did not create nor can men change. The latter are objects of science, the former are works of art. Although there may exist many curves which are at first difficult to categorize in this way, a practical rule for distinguishing them is strongly suggested in this passage:

> There is a great difference between the works of man and the works of God, that the same minute and searching investigation which displays the defects and imperfections of the one, brings out also the beauties of the other. If the most finely polished needle on which the art of man has been expended be subjected to a microscope, many inequalities, much roughness and clumsiness, will be seen. But if the microscope be brought to bear on the flowers of the field, no such result appears. Instead of their beauty diminishing, new beauties and still more delicate, that have escaped the naked eye are forthwith discovered.[21]

[21] Hislop, *The Two Babylons*, p. 1.

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