One final thought. God's habit of punishing wrongdoers is for the sake of the wrongdoer, to lead him to perfection. Is there any evidence that Moses has grown more perfect through this punishment? I cannot defend this strongly now, but it seems to me that the Book of Deuteronomy is evidence that he did. It is a book of lofty spiritual doctrine that transcends that of the 'first law' given on Sinai. It is here that we find the command to "love the LORD your God with all your heart, and with all your soul, and with all your might" (Dt. 6:5); and Christ uses its teaching to turn away the temptations of the devil (Mt. 4:4-10). There is also this text, which I think shows growth in Moses. As he is recounting the Lord's watchful care of them during the Exodus, he says: "And the LORD commanded us to do all these statutes, to fear the LORD our God, for our good *always*, that he might preserve us alive, as at this day." (Dt. 6:24, emphasis added) In his confession that the commands of the Lord were "for our good always," it seems that he is acknowledging that God is worthy of perfect love and obedience in a way that his actions earlier did not. Finally, there is Moses' appearance at the Transfiguration, together with Elijah, which may point to his bodily assumption into Heaven.[8] Thus, though his fault may have kept him from entering the Promised Land, the growth in love that it effected may have allowed him to enter the True Promised Land. O felix culpa!

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[8] This is also suggested by the text in the Letter of Jude (1:9), which alludes to a fight over the body of Moses between St. Michael and Satan. If the victor, St. Michael, were to return to heaven with it, which would explain why no one knows where Moses was buried, it would presumably be reunited with his soul.

# THE BEAUTY OF REASONING:
## CONSIDERATIONS ON BOOK V OF EUCLID'S *ELEMENTS*

### *Christopher O. Blum*

"What an exercise in logical precision it is," said John Henry Newman, "to understand and enunciate the proof of any of the more difficult propositions in Euclid."[1] Newman knew from first-hand experience the high value of the study of Euclid's *Elements of Geometry*. When he first arrived at Oxford, in the fall of 1817, he found himself faced with a demanding mathematics tutor who quizzed him about his preparation. "I believe, Sir, you never saw Euclid before?" Newman replied that he had "been over five books," but added "I could not say I knew them *perfect* by any means." The skeptical tutor asked Newman "what a point was, and what a line, and what a plane angle," and upon the student's correct answers, told him he should come "with the other gentlemen at 10 o'clock with the 4th, 5th, and 6th Books." "And today," Newman triumphantly told his mother, "after I had demonstrated a tough one out of the 5th Book, he told me I had done it very correctly." Indeed, Newman became so confident in his mastery of the material that when given a choice of texts on which to be examined at the end of the term, he picked "the 5th Book of Euclid, the hardest book of Euclid . . . the

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[1] John Henry Newman, "Discipline of Mind," in *The Idea of a University*, ed. Martin J. Svaglic (Notre Dame: University of Notre Dame Press, 1982), 378.

ratio of ratios book."[2] He was surely right to see Book V of Euclid's *Elements* as a worthy challenge. The book treats the theory of ratio and proportion in the abstract, that is, apart from the geometrical figures that the first four books have examined. And the abstraction poses serious barriers to our progress in understanding, because the intellectual custom of our age does not dispose us readily to appreciate the beauty of reasoning, which is the chief and almost the sole beauty of Book V.

Book V may be called "the hardest Book of Euclid" for many reasons. Some of the difficulties are with respect to the dispositions of the student. First, by leaving behind the triangles, parallelograms, and circles of the first four books, Book V marks an abrupt caesura in the unfolding narrative of the art of geometry. The student no longer has the satisfaction of seeing a step-wise and cumulative gain in knowledge that is in continuity from the initial definitions of Book I. This discontinuity is unsettling, and the unsettled student finds arduous study more trying. A second difficulty caused by the abstraction from figures is that the student's desire to learn is potentially sapped. There is a dryness to the theorems of Book V from which the previous books do not suffer, for if it is sometimes difficult to muster up interest in parallelograms or triangles, then Book V's bare lines that stand for any magnitude whatever will seem most unappetizing indeed. The third difficulty with respect to our dispositions is the most serious, for it concerns the change in the quality of the study that is required. Our inherited and perhaps even connatural knowledge of simple geometrical figures such as triangles allows many students after only a few moments of pondering a diagram to guess the crux of a theorem in the first four books. Not so in Book V. Cleverness must be checked at the door,

for the theorems are simply too far removed from common experience to admit of solution by guess-work.

Two further difficulties are intrinsic to Book V, that is, they arise from its proper subject matter. First is its new terminology. Because the theorems consider the truths of ratio and proportion in abstraction from figures, they must use a language that applies equally to different kinds of figures. The terms introduced in Book V, therefore, are general terms, that, in the English language, sound much alike: magnitude, multiple, equimultiple. The new terminology, moreover, is uncompromising, for to master Book V the student must even employ prepositions with care and precision: 'be a multiple *of*' signifies one thing and 'have a ratio *to*' another. The essential challenge of Book V, however, lies in the nature of the reasoning it involves, which is at once spare and complex. It is spare because the syllogism is unveiled in all of the grandeur of its lucidity; it is complex because the major premise of the most important syllogisms in Book V is its daunting 5th definition:

> Magnitudes are said to be *in the same ratio*, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.[3]

The simplicity and fertility of this definition are not immediately apparent. And yet in his history of ancient geometry, Proclus credited Euclid with "systematizing many of the theorems of Eudoxus," the original author of the theory of ratio

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[2] Newman's letter of November 13, 1817 to his mother, in *The Letters and Diaries of John Henry Newman*, ed. Ian Ker and Thomas Gornall, S.J., volume I (Oxford: Clarendon Press, 1978), 44–45.

[3] All quotations from Euclid's *Elements* are from the standard translation by Sir Thomas L. Heath, from the second edition published by the Cambridge University Press (1926) and now widely accessible in the series the *Great Books of the Western World*, as well as in editions from Dover Publications and Green Lion Press.

and proportion.[4] If, then, Euclid's great achievement was to have brought order and clarity to the study of geometry, it is worth assuming that a definition such as this one is in fact optimal. Our object here is to come to a better understanding of the place of the 5th definition in the science of geometry.[5]

In order to appreciate the beauty of the reasoning of Book V, we must first gain a general understanding of the subject of Books V & VI, the science of ratio and proportion, then explore the 5th definition of Book V and its function in demonstration, and, finally, to ask what Book V reveals, that is, how it paves the way for the investigation of ratio and proportion in geometrical figures in Book VI and beyond.

### Ratio & Proportion

The shift between Books IV and V of the *Elements* may be understood as a change from the study of equality to that of inequality. The shift is somewhat of a shock to the student, for it is equality that makes quantity intelligible to us, as when we measure something. We come to know the length of a table-top by discovering that it is equal to so many "feet," and the quantity "foot" is known to us both from long use and because it relates to our immediate experience of distance by walking. If two (or more) quantities were simply unequal to one another, we could say little more than "this is greater than that," or, at best, "this is much greater than that," and a reasoned-out account of their differences would be unnecessary. Books V and VI of the *Elements*, therefore, do not study unequal things as such, but unequal things that have certain

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equal relations among one another, as in "this is twice that," which is to say "this is equal to two of those." The subject of Books V and VI, then, is the kind of equality that is found among unequal things, that is, the equality of ratios, otherwise known as proportion.

The terms ratio and proportion can be more easily understood if we attend to Euclid's usage of them. "Ratio" is a carrying-over, through the Latin, of the Greek term *logos*, a very rich word, generally meaning either "speech," or "reason." In the context of Books V and VI of the *Elements*, ratio means the account of "the relationship in respect of size between two magnitudes of the same kind" (V. definition 3). Proportion is our English term for Euclid's *analogia*, which literally means "upon an account," that is, "according to an account." Euclid says, with admirable concision, "let magnitudes which have the same ratio be called proportional." It may be helpful, then, to think of proportional magnitudes as unequal or disparate magnitudes about whose size we are able to give an account that tells us more than that they are simply unequal.

An elementary understanding of proportion is gained during childhood, and chiefly from our parents. About the age of three, children begin keenly to notice inequality, especially when being served certain kinds of foods, and express their discovery in plaintive terms: "Why does Johnny have more ice cream than I do?" The answer, of course, is that Johnny is older. To explain that Johnny deserves more ice cream because he is older is to employ an elementary understanding of proportion. The proportion, in this case, sets unequal ages and unequal requirements for a healthy diet into a relation of equality, a sameness of ratio. A scant quarter-cup of ice cream, say, is an appropriate reward for a three-year-old who has eaten his broccoli, while half of a cup suits a nine-year-old (who has, presumably, eaten more broccoli).

As we mature, we begin to reflect upon proportion in connection with questions of justice. Like the three-year-old, men

and women seem to be initially inclined to interpret justice as a matter of strict equality. Consider the example of a high-school graduate wondering why her elder brother received a larger gift at his graduation from college than she did for finishing high school. The appropriate answer will repose upon the truth that justice is not a matter of strict equality, but instead, as Aristotle explains, "a species of the proportionate."[6] The just, in this case, is a certain sameness of relation, or ratio, that sets the rewards of the two siblings in proportion to their merits. As college required a more arduous period of study than did high school, it is appropriate for the elder brother to receive a proportionally larger gift. "The just, therefore, involves at least four terms; for the persons for whom it is in fact just are two, and the things in which it is manifested, the object distributed, are two."[7] In other words, the ratio of the merit of the first to the merit of the second is equal to the ratio of the first reward to the second reward.

Just as in the case of justice, so also in the study of geometrical figures: strict equality is understood before sameness of ratio. So, for instance, in Book I the equality of triangles is considered in three different theorems, which are commonly referred to as "side-angle-side" (I.4), "side-side-side" (I.8), and "angle-side-angle" (I.26). In each case, the theorem examines the conditions under which we may conclude that two triangles are equal in every way, that is, both in their shape and their size. In Book VI of the *Elements*, then, comes the consideration of triangles which have the same shape but *not* the same size, in theorem VI.4, in which it is demonstrated that "in equiangular triangles, the sides about the equal angles are proportional." The theorem is particularly satisfying because it accords with our expectations. We might very well guess that the sides of equiangular triangles are proportional and that the triangles are therefore similar. What VI.4 does for us

is to render that similarity fully known, that is, both explicit and the result of reasoning based on prior principles. We are able to conclude with certainty, therefore, that though the six lengths are disparate, they can be understood in terms of a kind of equality, the sameness of ratio, either about the given angles, or, alternately, as the corresponding sides subtending equal angles.

The science of proportion in Book VI extends and deepens the treatment of geometrical figures contained in the first four books of the *Elements*. Consider theorem VI.31: "In right-angled triangles, the figure on the side subtending the right angle is equal to the sum of the similar and similarly described figures on the sides containing the right angle." The resonance of this proposition to I.47, the Pythagorean theorem, is apparent. VI.31 states in general terms what I.47 says specifically about squares. In both cases, what is being said is that the sides of right triangles have a particular kind of relationship among them, not one of equality, but of equality in the squares or similar rectilinear figures constructed upon them. Proclus was so impressed by VI.31 that he saw it as proof of Euclid's superiority to Pythagoras: "though I marvel at those who first noted the truth of this theorem [I.47], I admire more the author of the *Elements*, not only for the very lucid proof by which he made it fast, but also because in the sixth book he laid hold of a theorem even more general than this and secured it by irrefutable scientific arguments."[8]

A still more significant example of the beautiful truths revealed by the study of ratio and proportion is theorem VI.30: "To cut a given finite straight line in extreme and mean ratio." AB is cut at E such that BE:EA::EA:AB.[9] Along the way, the rectangular parallelogram CD is constructed so as to be

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[6] *Ethics* V.3 at 1131a30.
[7] *Ethics* V.3 at 1131a18–21.

[8] Proclus, *Commentary*, trans. Morrow, 338.
[9] The notation A:B::C:D is to be understood this way: "A stands to B in the same ratio that C stands to D," or, more simply, "A is to B as C is to D."

equal to the square on AB; by subtraction, then, the square AE is shown to be equal to the rectangle AB,BE. This same result had been earlier accomplished in II.11, by means of the Pythagorean theorem. Here the more supple construction of VI.29[10] allows for an extremely economical path to the same result. Then the science of proportion provides a deeper, more complete understanding of what had earlier been learned in part about the same construction. For in II.11, it was shown that a line had been cut so as to make the square on one of the parts equal to the rectangle contained by the whole and the remaining part. The original line, however, was not plainly related to the two parts other than by being one of the sides of the rectangle. With VI.30, however, the fate of the given line is clearly understood: it has been cut in extreme and mean ratio, that is, the smaller part stands to the larger part in the same ratio that the larger part stands to the whole. What this in turn signifies Euclid does not say, for to do so would have taken him beyond the boundaries of Book VI. We can, however, see by comparing VI.30 with VI.9 that there is something grand at stake. For this is no ordinary "prescribed part," it is, in fact, an incommensurable or irrational ratio, and a special one, the Golden Ratio or Golden Section, so fruitful in works of art both human and divine, and, as we shall later see, in the science of geometrical solids.

## The Definition of Sameness of Ratio

Having seen that the science of ratio and proportion adds breadth and depth to the study of geometrical figures, we may now turn to the principles of the science itself. As in each of the first four books of the *Elements*, Book V begins with definitions. Here, the 5th definition is the first to pose a problem. Why is it so complex? Why must equimultiples

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[10] "To apply to a given line a parallelogram equal to a given rectilineal figure and exceeding by a parallelogram similar to a given parallelogram."

be examined in order to determine whether magnitudes are proportional? For in Book VII of the *Elements*, the first of Euclid's books on arithmetic, the definition of proportion is much more simple (VII. definition 20): "Numbers are proportional when the 1st is the same multiple, or the same part, or the same parts of the 2nd that the 3rd is of the 4th." Thanks to the facility with numbers we gain in grammar school, we readily perceive proportion in them, as in the case of the following examples.

4:2::8:4    [1st is same *multiple* of 2nd that 3rd is of 4th]
2:4::6:12   [1st is same *part* of 2nd that 3rd is of 4th]
3:4::6:8    [1st is same *parts* of 2nd that 3rd is of 4th]

A number, as Euclid teaches, is "a multitude composed of units" (VII. definition 2). All numbers have the unit as a common part or measure. Euclid's definition of proportion, therefore, is applicable to every number. Geometrical figures, however, are made of continuous quantity rather than discrete quantity. There is no "unit" in continuous quantity, and it is possible for two magnitudes to have no common measure, to be incommensurable. That, for instance, the side and the diagonal of a square have no common measure was well-known by the time of Plato.[11] It is because there are incommensurable magnitudes that a more general theory of ratio and proportion is necessary, and, with it, the taking and comparing of equimultiples. To our age, the notion of incommensurability has been lost, thanks to the custom of dissolving numbers into integers, mere place-holders for use in equations. The typical student today takes for granted the existence of negative numbers and repeating decimals, and is likely to be somewhat mystified by Euclid's theory of proportion as a result.

The 5th definition of Book V understands proportion in terms of its proper effect. Proportion in geometrical figures,

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[11] See Thomas Heath, *A History of Greek Mathematics, Volume I: From Thales to Euclid* (1921; Mineola, New York: Dover, 1981), 154–57.

or magnitudes, cannot be immediately judged in terms of common parts or a common measure. But even incommensurable magnitudes are capable of exceeding one another, which is why the 5th definition investigates the equality or inequality of equimultiples. Proportional magnitudes, then, are those whose equimultiples exceed, equal, or fall short of one another in the order set down in the 5th definition, that is, when equimultiples have been taken of the 1st and 3rd magnitudes and others of the 2nd and 4th, then the multiple of the 1st term is compared to that of the 2nd and that of the 3rd to that of the 4th. The investigation of the equimultiples discloses whether the original magnitudes are proportional. The 5th definition, therefore, involves a kind of reasoning from the effect back to the cause.

In coming to appreciate the 5th definition, it helps to consider the 7th definition of Book V in comparison. "When, of the equimultiples, the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to have a *greater ratio* to the second than the third has to the fourth." Here is an example of the 7th definition in use:

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A  ++++++++++        E  ++++++++++++++++++++
B  ++++++            G  +++++++++++++++
C  +++++++           F  ++++++++++++++
D  +++++             H  ++++++++++++++++
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A, B, C, and D are the original magnitudes. E and F are equimultiples of the 1st and 3rd magnitudes, that is, of A and C; each is double the original magnitude. G and H are other, chance equimultiples of the 2nd and 4th magnitudes, that is, of B and D; they happen to be three times the original magnitudes. G and H are called equimultiples because they are the product of the original magnitudes taken the same number of times. They are called "chance" equimultiples not because they happen to be the *same* multiple, but because the number

by which the original magnitudes were multiplied happened to be three. When we compare the equimultiples, we see that the multiple of the first, E, exceeds that of the second, G, while that of the third, F, falls short of that of the fourth, H. We may conclude, therefore, that the ratio of A to B is greater than that of C to D (A:B > C:D). And when we inspect the original magnitudes, we see that the conclusion plainly fits with what appears to be the case.

What if the original magnitudes had been in the same ratio? Let us recall the wording of the 5th definition: "Magnitudes are said to be *in the same ratio*, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order." The following example differs markedly from the previous one.

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A  +++++++           E  +++++++++++++++++++++
B  ++++++            G  +++++++++++++++++++++
C  ++++              F  ++++++++++++
D  +++               H  ++++++++++++
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We have again taken equimultiples of the 1st and 3rd terms, by taking A and C each three times to generate E and F. We have taken other, chance equimultiples of the 2nd and 4th terms. G and H are each four times B and D. In this case, E is equal to G while F is equal to H. It is plain that if A and C had each been taken four times instead of three, then E would exceed G while F would exceed H. Or, again, if A and C had each been taken twice instead of three times, then E would fall short of G and F would fall short of H. Only when the original magnitudes are in the same ratio will the equimultiples *always* behave in the proper fashion. If, however, the ratio of the 1st to the 2nd had been different—say, greater—than that of the 3rd to the 4th, then there would

be some combination of equimultiples that does not behave according to the dictates of the 5th definition, as we saw in the case of the first example above.

Of the twenty-five theorems in Book V, only ten require the investigation of equimultiples. The first of these is V.4: "If a first magnitude have to a second the same ratio as a third to a fourth, any equimultiples whatever of the first and third will also have the same ratio to any equimultiples whatever of the second and fourth respectively, taken in corresponding order." Like each of the first six propositions in Book V, this theorem examines a characteristic property of equimultiples. These six theorems provide the tools for the consideration of equimultiples in the later propositions that treat proportion itself, theorems such as V.12, which makes use of V.1, V.17, which makes use of both V.1 and V.2, and V.22, which employs V.4 itself. Let us consider V.4 in detail.

The demonstration begins by laying down the four original magnitudes, which by hypothesis are in the same ratio (A:B::C:D). Then chance equimultiples are taken of the 1st and 3rd terms, and others of the 2nd and 4th terms.

A ++++++++    C ++++++++++
B +++    D ++++
E +++++++++++++    F ++++++++++++++
G +++++++++++    H ++++++++++++

In this case, E and F are each double A and C respectively, while G and H are each triple B and D. E, F, G, and H are now to be considered in themselves. Do they stand in the same ratio to one another, that is, does E stand to G in the same ratio that F stands to H (E:G::F:H)? Upon inspection, it appears likely that the magnitudes are proportional, but the geometer seeks certitude, not likelihood. The only way to know whether they are indeed proportional is to investigate the equimultiples generated from them. The next step in the demonstration, therefore, is once again to take chance equi-

multiples of the 1st and 3rd terms, and others of the 2nd and 4th.

K +++++++++++++++++++++++    L ++++++++++++++++++++++++++++
M ++++++++++++++++++++++++    N +++++++++++++++++++++++++++++

As it happens, K and L are each double E and F, while M and N are each triple G and H. We must consider K, L, M, and N now as equimultiples, that is, as magnitudes that either exceed, equal, or fall short of one another taken in corresponding order. We are faced with a question: Is it the case that when K exceeds M, L also exceeds N, and that when K is equal to M, L is equal to N, and, finally, that when K falls short of M, L also falls short of N?

Such a question seems difficult to answer. How are we to know anything about the relative sizes of K, L, M, and N? What we do know, however, is that the magnitudes from which these equimultiples were generated were themselves equimultiples of four proportional magnitudes. The next step in the demonstration, therefore, is to relate K, L, M, and N back to the original, proportional magnitudes A, B, C, and D. This task may be accomplished thanks to the preceding theorem, V.3. It is not our business here to examine the demonstration of that theorem. It is enough for us to see that our problem is solved by the truth revealed by V.3: "If a first magnitude be the same multiple of a second that a third is of a fourth, and if equimultiples be taken of the first and third, then also *ex aequali* the magnitudes taken will be equimultiples respectively, the one of the second, and the other of the fourth." *Ex aequali* means "from the equal thing;" the connotation is that we are able to conclude equality "from the equal thing" that lies between the two objects in question, as if we were saying, in this case, "equal multiples of equal multiples are also equal multiples of the original magnitudes."

Returning to our example, we may now apply V.3 to our equimultiples. A first magnitude, E, is the same multiple of a

second, A, that a third, F, is of a fourth, C, and a fifth magnitude, K, is the same multiple of the first, E, that a sixth, L, is of the third, F, therefore the fifth and sixth magnitudes are also equimultiples respectively of the second and fourth. That is to say that K and L are equimultiples of A and C. For the same reason, M and N are equimultiples of B and D.

Now that we have successfully identified K, L, M, and N as equimultiples of the original magnitudes, which we know to be proportional, we may invoke the 5th definition. Since K and L are equimultiples of A and C, while M and N are other, chance equimultiples of B and D, while A:B::C:D, the following relationships of equality and inequality necessarily follow:

when K > M, also L > M;
when K = M, also L = N; and,
when K < M, also L < N.

In the example above, K and L each fall short of M and N respectively.

Having discovered the relationships that obtain among the equimultiples K, L, M, and N, we may now look back to the magnitudes from which they were generated, E, F, G, and H. Our reasoning, again, follows the course of the 5th definition, but this time in the opposite direction. Since it is the case that

when K > M, also L > N;
when K = M, also L = N; and,
when K < M, also L < N,

while K and L are equimultiples of E and F and M and N are other, chance equimultiples of G and H, then it follows that the original magnitudes are in the same ratio, the first standing to the second as the third stands to the fourth, that is, E:G::F:H. We have now completed the demonstration, for we have shown, with the help of the 5th definition, that if a first magnitude have to a second the same ratio as a third to a fourth, any equimultiples whatever of the first and third will

also have the same ratio to any equimultiples whatever of the second and fourth respectively, taken in corresponding order.

In order to dispel a possible point of confusion, we should note that the ratio of E:G (and so also of F:H) is not necessarily the same ratio as A:B (or C:D). We can appreciate this fact better if we supply numbers to the magnitudes we used above. Recall that E and F were each double A and C, while G and H were each triple B and D:

A = 8    B = 4    C = 10    D = 5
E = 16   G = 12   F = 20    H = 15

The ratio of A:B, therefore is 8:4, or 2:1, while that of E:G is 16:12, that is, 4:3. When we inspect the numbers, then, it is plain that A:B::C:D and also that E:G::F:H, but that the two sets of four magnitudes are not necessarily in the same ratio with one another.

Now that we have carefully followed the demonstration of a theorem reposing upon the definition of sameness of ratio, we should stop to consider the form of reasoning employed in the demonstration. The 5th definition, though itself complex and even cumbersome, is employed in the simplest of syllogisms, those of the first figure. The identity of the major term is plain: "in the same ratio," or, simply "proportional." The middle term, however, cannot be so neatly expressed, for it is the proper relationship of the equimultiples generated from the original magnitudes. In other words, the middle term is the entire second half of the 5th definition: "when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order." For convenience, then, we may express the middle term as *magnitudes that generate equimultiples with the proper characteristics*. The minor term can be thought of in two ways. As a particular affirmation, the minor term is simply "these four magnitudes." As a universal

affirmation, the minor term will vary according to what is proposed in the given theorem. In the case of V.4, the minor term would be "the equimultiples of the 1st and 3rd and 2nd and 4th terms, respectively, of magnitudes that are in the same ratio." The concluding syllogism in V.4, then, would read as follows, stated universally:

I. *Magnitudes that generate equimultiples with the proper characteristics* are proportional.

II. Magnitudes that are the equimultiples of the 1st and 3rd and 2nd and 4th terms, respectively, of magnitudes that are in the same ratio are *magnitudes that generate equimultiples with the proper characteristics*.

Conclusion: Magnitudes that are the equimultiples of the 1st and 3rd and 2nd and 4th terms, respectively, of magnitudes that are in the same ratio are proportional.

Or, with respect to our example above:

I. *Magnitudes that generate equimultiples with the proper characteristics* are proportional.

II. E, G, F, and H are *magnitudes that generate equimultiples with the proper characteristics*.

Conclusion: E:G::F:H

Midway through V.4 it was necessary to employ the 5th definition in reverse, that is, to place the relations of the equimultiples as the major term, with "proportional" as the middle term:

I. Proportional magnitudes are *magnitudes that generate equimultiples with the proper characteristics*.

II. These magnitudes (in our example: A, B, C, and D) are proportional.

Conclusion: These magnitudes are *magnitudes that generate equimultiples with the proper characteristics*. (In our example, K>M, L>N; K=M, L=M; K<M, L<N.)

The essential structure of the reasoning of the crucial propositions in Book V, therefore, is to move through first-figure syllogisms employing the 5th definition in the major premise.[12]

### The Achievement of Book V

Having gained a general sense of the science of ratio and proportion and having examined the kind of reasoning that undergirds it in the central theorems of Book V, it remains to be seen how Book V makes possible the study of ratio and proportion in geometrical figures in Book VI and, beyond it, in Books X through XIII.

Book VI begins with a proposition that is essentially the same in structure as the crucial theorems of Book V, in that VI.1 reposes upon the Book V's 5th definition. VI.1 proves that triangles and parallelograms under the same height are to one another as their bases, and it proceeds by the investigation of the equimultiples of the bases and the triangles. What VI.1 adds to the ordinary structure of the theorems that employ the 5th definition is the geometrical principle that triangles on equal bases and in the same parallels are equal (I.38). This truth secures the proper relation of the equimultiples, namely, when the multiple of the first base exceeds that of the second, then the multiple of the first triangle will also exceed that of the second, and if the bases are equal, so are the triangles, and if less, less. As the entire subject of Book VI rests upon this theorem as its foundation, it is plain that Book VI also

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[12] Additional insight could be gleaned by the comparison of Euclid's practice to the doctrine of Aristotle's *Posterior Analytics*, which comparison, however, lies beyond the scope of this introduction to Book V.

reposes upon the 5th definition and the form of reasoning that it involves.

A second way of perceiving the necessity of Book V for the science of ratio and proportion in geometrical figures is to examine the use of its theorems in the demonstrations of Book VI. What such an inquiry reveals is that five of the theorems of Book V are particularly crucial: V.7 together with its converse V.9, V.11, V.16, and V.22. Of these, the most frequently employed is V.11: "ratios which are the same with the same ratio are the same with one another." V.11 is directly used in eleven of the thirty-three propositions in Book VI. Also significant is V.24, which, in effect, allows the addition of proportions.[13] Although V.24 is used only once in Book VI, the theorem for which it is necessary is that generalized restatement of the Pythagorean theorem (VI.31) that is one of the book's crowning achievements.

The most compelling witness to the importance of Book V, however, is to be gained by considering its relation to the most significant theorem of Book VI, the cutting of a line in extreme and mean ratio (VI.30). To cut a line AB such that the smaller part stands to the larger in the same ratio that the larger stands to the whole (BE:EA::EA:BA) is that Golden Section so wonderfully employed in such buildings as the Parthenon and Amiens Cathedral and also in the creation of the human body. What is more, the Golden Section is the linch-pin that holds together the narrative structure of Euclid's Elements. As Proclus noted, Euclid, as a follower of Plato, conceived of "the goal of the Elements as a whole to be the construction of the so-called Platonic figures," those five solids, made of equiangular and equilateral plane figures and inscribed in spheres: the pyramid, the square, the octagon, the icosahedron, and the dodecagon.[14] To create the last two of

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[13] Expressed in symbols, V.24 proves that if A:B::C:D and E:B::F:D then A+E:B::C+F:D.

[14] Proclus, Commentary, trans. Morrow, 57.

these figures, the twenty-sided and the twelve-sided, requires the ability to cut a line in extreme and mean ratio as well as the knowledge of the relations among the figures that may be generated from the parts of a line so cut. The importance of VI.30 to the whole of Euclid's Elements, then, can scarcely be exaggerated. It is as good a candidate as any for the honor of most worthy theorem in the entire treatise. VI.30 is also a reliable gauge of the mastery of the first six Books of the Elements, for to be able to complete VI.30 requires not only the knowledge of the greater part of Book VI, but also of 18 of the 25 theorems contained in Book V.

## A Concluding Reflection

As founding rector of the Catholic University of Ireland, John Henry Newman's essential task was to create an educational institution that would both be faithful to the Catholic intellectual tradition and well-suited to serve the needs of its constituents. His Discourses on University Teaching, his lectures on University Subjects addressed to Members of the Catholic University, and essays on the Rise and Progress of Universities remain as testimony to his principled approach to this great challenge.[15] It has perhaps not been sufficiently noted that Newman not only defended the role of theology as the queen of the sciences, but also fought the tendency of educators in his age to abandon those elementary studies, such as Euclid's Elements, necessary for gaining what he called "discipline of mind." Indeed, he insisted that liberal education ought to be a training in reasoning and accuracy of thought. Thus he held that "the first step in intellectual training" was to "impress upon [the] mind the idea of science, method, order, principle, and system."[16] And he warned, repeatedly, of the tendency among

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[15] The first two works mentioned have long been published together under the title The Idea of a University.

[16] From Newman's preface to his Discourses [1852], in Idea of a University, ed. Svaglic, xliv.

the youth toward "mental restlessness and curiosity," which was commonly joined to a distaste for mathematics. This disinclination he viewed in a sharply negative light: "[it] only means that they do not like application, they do not like attention, they shrink from the effort and labour of thinking, and the process of true intellectual gymnastics."[17] Yet Newman's educational vision was not the traditionalism of a crotchety old man. He held a compelling positive ideal, the "perfection of the Intellect" that resulted from the arduous training and careful study he recommended. For the mind disciplined by liberal education, he explained, attains a "clear, calm, accurate vision and comprehension of all things," and,

> is almost prophetic from its knowledge of history; it is almost heart-searching from its knowledge of human nature; it has almost supernatural charity from its freedom from littleness and prejudice; it has almost the repose of faith, because nothing can startle it; it has almost the beauty and harmony of heavenly contemplation, so intimate is it with the eternal order of things and the music of the spheres.[18]

How fitting it was that Newman concluded his impressionistic description of the educated mind on a Pythagorean note. His own mind had been formed not only by the Latin and Greek Classics and the works of Aristotle, but also by Euclid's *Elements of Geometry*. At the heart of that great book, in the theory of ratio and proportion of Book V, one may indeed catch a glimpse of the beauty of reasoning.

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[17] "Elementary Studies," in *Idea of a University*, 255–56.
[18] Newman, Discourse VI, in *Idea of a University*, 105.

# ANIMALS, INERTIA, AND THE CONCEPT OF FORCE[1]

### *Dr. Sean Collins*

A striking thing you discover when you read Copernicus is that his theory was not complicated. All Copernicus wanted to show was that the planets go around the sun and not around the earth. Not only is this not very complicated, it wasn't even new when Copernicus proposed it. Long before Copernicus, the ancient Greek philosopher Aristarchus had already proposed it.

Still, it wasn't easy for Copernicus to convince his contemporaries to take this idea seriously. But then, this is not really surprising. The most important ideas are often not complicated. Their importance comes from the fact that they are *seminal* ideas, which serve as principles—not only because many other things follow from them, but also because they *form our vision* of the world, our way of looking and our way of seeking out the truth—and this especially is why they are not easily accepted.

Today I want to discuss a few such ideas. The thesis I shall propose today began years ago for me as a question. One day during my graduate school years, I was talking with colleagues

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