What Is a Limit?

John Francis Nieto

1. Among the paradigmatic instances of the method of limits is the approach of polygons to a circle. The circle is drawn. A polygon, preferably a square or triangle, is inscribed. The arcs of the circle are bisected and lines are drawn from the points of bisection to the two endpoints of each arc. A regular figure of twice as many sides results. A hexagon. Again, the arcs are bisected and lines produced to their endpoints. A dodecagon. But this bisection and production of figures can be repeated indefinitely. The circle does not determine an end to the process. Yet more wonderful, there is no part of the circle, neither a set of lunules, nor an area distributed among ever many more dwindling lunules, that will remain untouched by the process. While by this method, the polygons will never consume the entire circle, they are insatiable, taking more and more of the circle unto themselves. Though they can never reach beyond the circle, nor even attain it, the areas contained by polygons can come closer to the area of the circle than by any difference however small.

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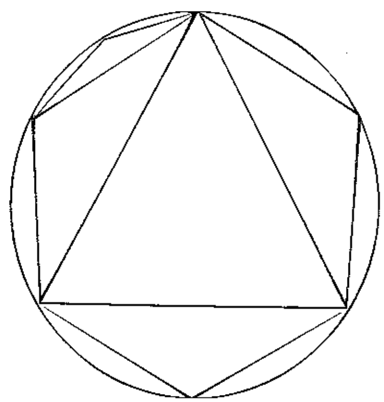


2. This method of producing polygons begins in the imagination, which makes a "movement" of these figures. A triangle "becomes" a hexagon, which "becomes" a dodecagon, which "becomes" a twenty-four-sided figure. Once one recognizes that the process is boringly interminable, the imagination races ahead. An "ambiguously" sided figure represents the indefinitely many figures that remain. For the mind perceives the unceasing ability of constructing new polygons. This "motion", insofar as it involves one figure becoming another, is unending. But it does have a limit. This limit is no polygon, if polygons all possess finitely-many sides. Rather, the limit of all these polygons contained by finitely-many sides is a circle. So Galileo defines the circle as an infinitely-sided polygon.

3. Now, two difficulties have appeared in this examination of the approach of polygons to the circle. One involves motion and its end, the other regards the definition or conception of the circle as an infinitely-sided polygon. These will be considered two essential aspects of limits which this paper will examine briefly. The very word "approach" suggests a movement which is to some "limit", defined by Aristotle as

"the first outside which one finds none of the thing and the first within which one can find all." (*Metaphysics* 1022a4–5) Again, many things can be understood as limits. A tangent is the limit of secant lines, the circle is the limit of inscribed and circumscribed polygons, the parabola is the limit of inscribed vertical triangles. To describe the circle as a limit is perhaps to say what is less cautiously and with less truth said when Galileo calls it an infinitely-sided polygon. Though not all instances of the method of limits go so far as to define or describe the limit and limited in terms of each other, the possibility and inclination to do so is among the most difficult and most important aspects of the method.

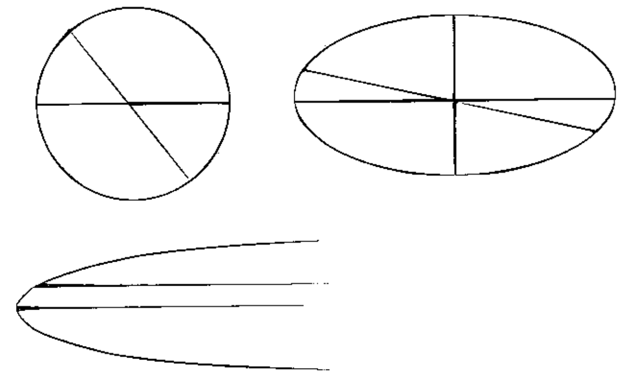
4. Examining these two aspects of the limit, one recognizes something else. The understanding of the limit as a limit, its definition in terms of what approaches it (or vice-versa), is the purpose of this motion or approach. The motion of the approaching polygons exists for the sake of considering the circle as their limit. To speak boldly, the understanding of the circle as a limit is the result and the conclusion of the motion. While no end is in sight to the one imagining the inscription of polygon after polygon, this understanding of the circle as their limit arises. In fact, the motion is no longer produced upon arriving at such an understanding.

5. Now, motion is determined by its term, and the species of motions are divided by their terms. So becoming hot differs from becoming cold. So, the nature of the movement that occurs in the approach to the limit should be more clear through a grasp of the kind of consideration in which it terminates.

6. A penetrating example of this sort of consideration is found in a short unfinished treatise by Blaise Pascal on the "conic sections", that is, the various lines which are the intersections of cones and planes. Pascal presents us with a new account of the conic sections, unlike that handed down through Euclid and Apollonius. His method is uniquely fit for our examination, for it provides a view of the conic sections sim-

ilar to that attained by the method of limits—without the motion. In addition, Pascal offers an imaginable foundation for such a consideration. He has found a vantage point from which he can see the conic sections in a fashion analogous to their considerations as limits or limited.

7. To understand what Pascal achieves, some characteristics of the conic sections must be kept in mind. These are all proved by Apollonius in his "Euclidian" treatment of the cone. First, there are three sorts of conic section. A cone may be intersected by a plane to produce an hyperbola, a parabola, or an ellipse, allowing the last term to include the circle. Each of the conic sections possesses infinitely many diameters. These diameters are lines which cut the section such that all lines within the section and falling upon the diameter at a certain angle are bisected. From any point on each of the conic sections, a diameter can be drawn. In the ellipse and circle these lines are finite in length and all meet at the center of the figure, which is within the figure. The diameters of the parabola are all parallel to one another, and all can be produced indefinitely. So, they never meet, and the figure has no center. Finally, the hyperbola has diameters which can also be produced indefinitely, but these are neither parallel to one another nor do they converge upon one another within the section. They meet outside the section and are produced in that direction to become diameters of a mirroring hyperbola, that is the other half of the hyperbola. Pascal's consideration of two things are of particular interest: the relation between the points on the circumference of the circle and the points on the conic section and the relation between lines in the circle and the diameters of the parabola.



8. Now, little need be said about Pascal's generation of the cone. Like Apollonius and not like Euclid, he begins with a circle and a point outside the plane of this circle. He then produces a line from this point to a point on the circumference of the circle. This line may be produced indefinitely in either direction. The line is then rotated through the circumference of the circle, while its endpoint, the point outside the plane of the circle, remains fixed. So a cone of any angle is produced, having as its base the circle which served in its production, and as its vertex, the fixed point.

9. Pascal's novelty begins after the generation of the conic surface. Taking the cone, Pascal places his eye at the vertex of the cone. From the vertex he looks toward two things, the base of the cone, which is a circle, and a plane intersecting the cone. This plane intersects the cone in any position and upon it will be "painted a conic section, so it is referred to as a *planum de tabella* or *plan de tableau*." It might well be called the "easel" or the "plane of the painting", but, for simplicity's sake, will be called the "tableau".

10. Now, Pascal is not merely looking at the conic sections which appear on the tableau. Two things must be mentioned. First, Pascal generates the conic sections as the projections or appearances of the circle which is the base of the cone. This is

a Platonic generation in which difference results from matter, namely the tableau as it takes on various positions. The rôle of such generation in the method of limits must unfortunately be ignored in this lecture.

11. A second point, closely related to this first and of greater present importance, is that Pascal looks at the various conic sections through the circle. This image of the eye viewing the parabola through a circle is a tool for comprehension of the method of limits. For the purpose of the method, its result, is precisely this, to consider something through the definition of another. Why this is desirable can be put off until later. At present it suffices to see that Pascal has chanced upon the imaginable analogue of the sort of knowledge obtained by the method of limits. As noted above, the various conic sections are the appearances of circles on various tableaus. Examining Pascal's account of the diameters and points of the conic sections will show that such an imaginable foundation led him to an account of the figures through his understanding of the circle.

12. This may seem most appropriate. For all knowledge, including definitions, must be through what is more known. But the more known can be "in obliquo"—in a case, to speak grammatically. The circle is defined neither as a straight line nor as composition of them, rather it is defined with reference to them. So they appear in the Greek of Euclid and the Latin translations of his *Elements* in the genitive or ablative case. The use of the term 'limit' seems an attempt to do the same, that is to use 'polygon' indirectly to define the circle. Pascal's account, however, and that in some ways essential to the method of limits, allows that the circle be defined with the more known "in recto", in the nominative case—directly, to approximate. So the circle is a polygon of infinitely many sides and the conic sections are projected circles.

13. Viewing the conic sections as the projections or appearances of the circle, Pascal has little problem with the various ellipses. For any point on the circumference of the circle

there is a corresponding point on the ellipse and any straight line cutting the circumference of the circle appears as a finite straight line in the ellipse.

14. In the parabola and the hyperbola, this is not quite so. Though every point on these lines corresponds, through some position of the generating line, to a point on the base, not every point on the base will have a corresponding point on these two conics. One point of the circle does not appear on the parabola, two do not appear on the hyperbola's two branches.

15. One could merely attribute this to the deficiency of the matter. These would-be circles exist in a plane insufficient for the full manifestation of their nature. But Pascal is not satisfied with such an answer.

16. A second problem involves the appearances of the lines within the base upon the parabola. Any line cutting this circle's circumference will show up in the ellipse as a finite line. But in the parabola only some of the circle's lines appear as finite, namely those lines in the circle which *do not* have as an endpoint the point which does not show up. Quite reasonably those lines whose endpoint does not appear on the parabola themselves appear on the parabola as infinite lines. On the circle instances of these lines will splay across the circle and obviously all meet one another at the point without appearance. But the appearances of these lines within the parabola are not tied together by any point of intersection. Rather they run off in infinitum and parallel to one another. Poetically speaking, these lines pursue the appearance of their intersection in vain.

17. Overlooking the complex difficulties with the appearances of the straight lines in the circle within the hyperbola, we can sum up the deficiencies of the conics as appearances of the circle as follows: The parabola is missing a point, for lack of which its diameters are parallel and infinite. The hyperbola is missing two points.

18. Precisely the attempt to overcome these deficiencies illuminates Pascal's consideration of the conics through the

definition of the circle. Each of the deficiencies is overcome by the consideration of an infinite distance. In the corollary describing the generation of the parabola, Pascal says,

> It is manifest that all the points of the circumference project their images upon the plane of the conic section, at a finite distance, except for one point, which has no image but at an infinite distance.

Pascal offers two ways of describing the point. It does not have an image on the tableau. It has one at an infinite distance. The one is a dull statement of fact. There is no point on the parabola to be seen which corresponds to this particular point on the circumference of the circle, through which I am now looking. To then assert, the image exists at an infinite distance from this point is not merely to repeat oneself. Even if I insist that these "formulae" describe one and the same thing, I am not now merely repeating myself, having said: "There is no image, the image exists nowhere". Rather, I am stating something nearer to, "The image exists beyond the other projected points and beyond my power of sight, but it exists."

19. But how are the various diameters related to such an image? In the circle, the exemplars of these diameters are so many straight lines drawn from the "point without image" to the circumference of the circle. In the parabola, the images, the diameters, are all parallel, pursuing in infinitum an image of the point without image. But Pascal has introduced a "definition" of the "tending of straight lines", which overcomes this apparent lack of conformity.

> A straight line is said to tend to a point, which arrives at that point, if produced, and a straight line is said to be drawn or to tend to a point given at an infinite distance on another line which is parallel to it.

So each diameter of the parabola tends to some point on the other that stands at an infinite distance.

20. The use of such language may seem of little importance. Certainly there is neither an image of certain points nor a point

of intersection even though such language is used. No one asserts that the images of these points actually exist.

21. Yet this language is not used to convey the thought of these points as not existing. Such language is used so that the points on the circle can be viewed as existing in the conic, so that the conic can be seen through the circle and as a sort of circle. So the very notion of "tending" is toyed with. But, more importantly, parallel lines are looked at "through" lines which intersect, and understood through them. Perhaps even non-existence is reconsidered through the addition of an infinite distance, rather than negation, to the concept of existence.

22. Why does Pascal want to consider conics in this way? This seems an act of folly, to consider things through the definitions of other things, when their own are at hand. Things are obviously better known through their own definitions. What shall we prove about the circle through the definition of polygon?

23. Now, the knowledge obtained in this way is not superior as knowledge to knowledge obtained by the proper definitions of things. But the manner or mode of knowing is superior. For by this sort of consideration the means of our knowledge are reduced and unified. Pascal is not merely seeking to look at each of the conics through a circle, he is pursuing one means through which he can view them all, one concept in which he can consider the ellipse, parabola, *and* hyperbola, just as he can look at them all through the circle. Rather than considering each by its own conception, he can consider them all in one concept.

24. This is a more powerful union than the union of the various figures in their genus, namely conic section, or that of man and horse in animal. In their genus things are understood without the character proper to them, without the very difference which distinguishes them from others contained in the genus. Yet by the sort of consideration laid out above, things are seen through other concepts without losing their

peculiar nature. The parabola can be seen through the circle as a parabola, as possessing all that distinguishes it from a circle. Likewise the circle, considered as an infinitely-sided polygon, retains precisely what distinguishes it from all polygons. So this is clearly a different unity than that things have in universal concepts, in which something belonging to them, either the individuating matter or some difference, must be overlooked.

25. The profit of reducing and unifying our means of knowing becomes clear in a general way when one considers the divine mode of knowing. The poverty of our thought is clear enough from the fact that we must think "triangle", "figure having its exterior angle equal to its opposite interior angles", "figure having all angles equal to two right angles", and so on. Our concepts are many and varied because we know things imperfectly. The knowledge obtained by a single intelligible form must be expressed in many concepts. Clearly God has no such imperfection. But God not only does not need many concepts to express his knowledge of any one thing, he knows all things by a single intelligible form. As Saint Thomas says:

> . . . the intellect can certainly understand many things *per modum unius*, but not many *per modum multorum*. Now I mean *per modum unius vel multorum*, through one or many intelligible species. For the mode of any action follows the form which is the principle of the action. So whatever the intellect can understand under one species it can understand together. So God sees all things at once because he sees all things through one, which is his essence. . . .

So the reduction of our means of knowing, as Pascal has reduced the understanding of the various conics to that of the circle, produces a mode of knowing akin to the divine mode, in which all things are seen even in their particularity by a single form, the divine essence, and expressed in a single concept, the Word. (Recall that Pascal not only considers the conics through the circle. He also generates them through it.)

26. A similar reduction can be attained by the method of limits. The circle and its inscribed polygons can all be considered through the notion of polygon. Again, the straight line can be considered as a sort of circle, one of infinite diameter. The inability to demonstrate from such definitions shows such knowledge is not fit for humans. But the terrifying power over nature that such considerations allow adds weight to the claim that man has hit upon a sort of knowledge far above him.

27. Now the kind of consideration just discussed arises at the term of the movement or approach used in the method of limits. In its most extreme expression something comes to be its opposite. A polygon (having infinitely-many sides) becomes a circle. A circle (of infinite diameter) "becomes" a straight line. Again, the infinitely many instances of what is limited cannot be distinctly considered. If the 'movement' or approach depends upon actually considering each of these instances, it cannot actually reach its term. This is precisely Zeno's paradox.

28. But how is there any motion which can overcome the infinity of species and the contradiction in the term such that an extrinsic limit of the genus can be seen as if in the genus, albeit uncomfortably? A true movement is certainly impossible. Earlier the imagination seemed to compose this motion by lining up, one after another, the various species of the limited polygons. There appeared neither need nor ability to complete the lining up. Considering the order of a limited number of polygons, the intellect was able to determine the bounds beyond which this order could not pass. The only motions were the passing from one image to the next and finally to the limit and the mind's turning from consideration of one to that of the other, if either of these is a motion. The real movement that this seems most like is growth, which also occurs in 'stages', although these could not be infinite.

29. In fact, these motions, the passing from one image to the next, seem more like 'moments'. The new position of the

piece in a chess game, where the physical movement is accidental, brings about such a moment. The chess game itself is composed from the various 'set-ups' that arise from each 'move' as from certain 'moments'. Thus a game can be described quite exactly by reporting each 'move'. This sort of 'movement' is not continuous, but is 'composed' from these discrete moments. Each of the infinite members of the series that is ordered to the limit is such a moment. Leaving aside consideration of the various psychological acts necessary for the approach to the limit, it is clear that insofar as it is a motion the approach uses these members of the series as moments.

30. Nonetheless, the limit, the final moment, is like the term of a movement and so the language of motion is not without reason. The very possibility of 'conceiving' the circle as an infinitely-sided polygon reveals that it 'completes' the polygons or brings them to an end. If we imagine the species of polygon as so many parts of a whole, as in the calculus, the circle as the limit of this series is certainly "the first outside which one finds none of the thing and the first within which one can find all." (*Metaphysics* 1022a4–5)

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