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A NOTE ON PROPOSITION I, 41 OF APOLLONIUS' *ON CONIC SECTIONS*

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WHETHER the circle ought to be considered a species of the ellipse is a question debated by students of Apollonius' *On Conic Sections*. It is clear that Apollonius does not consider it to be so, and though many good reasons may be given to support his position, he yet excludes the circle by a seemingly arbitrary restriction on the way the cutting plane may lie with respect to the plane of the base circle: "If a cone is cut by a plane through its axis, and is also cut by another plane on the one hand meeting both sides of the axial triangle, and on the other extended neither parallel to the base nor subcontrariwise...let such a section be called an ellipse." (Prop. I,13).[1]

The reason why Apollonius had to include such a restriction is that the property proved of the ellipse in this, its defining proposition, belongs also to the circle. Proposition 13 proves that in the figure produced by the cut described above "any straight line which is drawn from the section of the cone to the diameter of the section parallel to the common section of the planes will equal in square some area applied to a straight line to which the diameter of the section has the ratio that the square on the straight line drawn from the cone's vertex to the triangle's base parallel to the section's diameter has to the rectangle contained by the intercepts of this straight line (on the base) from the sides of the triangle, an area having as breadth the straight line cut off on the diameter beginning from the section's vertex by this

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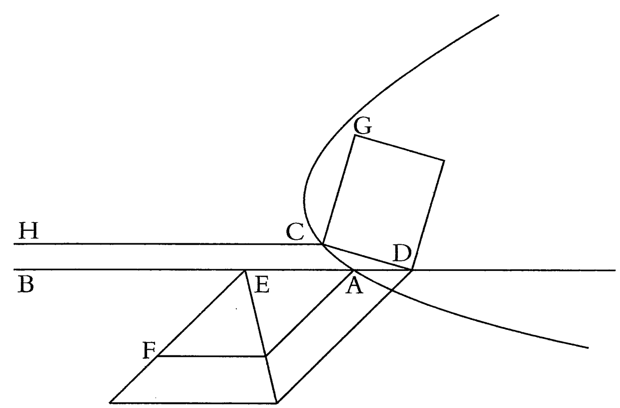
straight line from the section to the diameter, and deficient by a figure similar and similarly situated to the rectangle contained by the diameter and parameter." This property, as may easily be verified, belongs also to the subcontrary circle defined in Proposition I, 5 and hence to all circles. For the circle, the parameter (also called the upright side) is of the same length as the diameter (or transverse side).

Because of the equality of the parameter and the diameter of the circle, the property proven in Proposition 13 may be stated more simply: "If a line be dropped from the circumference of a circle perpendicular to a diameter, the square on the dropped line is equal to the rectangle contained by the segments of the diameter." This property of the circle, easily deduced from Euclid, *Elements* III, 33 and the porism to VI, 8, is used to prove the corresponding property of the ellipse. Since the property of the circle is included as a principle in the proof of Proposition 13, it is not surprising that it can also be extracted from the proposition.

Many of the propositions in Book I applying to the hyperbola and the ellipse are also enunciated for the circumference of the circle. It is natural to wonder in all these instances whether the circle's property is a simpler truth that we already know from Euclid. A particularly interesting case is Proposition 41.

The enunciation and setting out of this proposition are as follows:

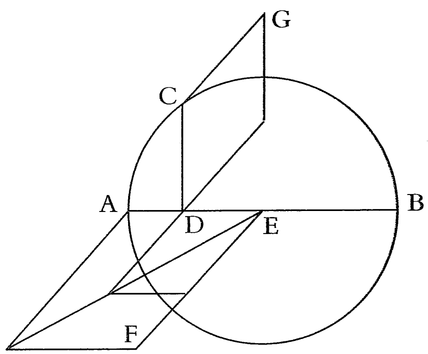
> If in an hyperbola or ellipse or circumference of a circle a straight line is dropped ordinatewise to the diameter, and equiangular parallelogrammic figures are described both on the ordinate and on the radius, and the ordinate side has to the remaining side of the figure the ratio compounded of the ratio of the radius to the remaining side of its figure, and of the ratio of the upright side of the section's figure to the transverse, then the figure on the straight line between the center and the ordinate, similar to the figure on the radius, is in the case of the hyperbola greater than the figure on the ordinate by the figure on the radius, and, in the case of the ellipse and circumference of a circle, together with the figure on the ordinate is equal to the figure on the radius.
>
> Let there be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line AB, and center the point E, and let the straight line CD be dropped ordinatewise, and on the straight lines EA and CD let the equiangular figures AF and DG be described, and let CD:CG comp. AE:EF, the upright: the transverse. I say that, with the figure on ED similar to AF, in the case of the hyperbola, figure on ED=AF+GD and in the case of the ellipse and circle, figure on ED+GD=AF.



As one might expect, the construction is simpler in the case of the circle. Let us go through the construction step by step.

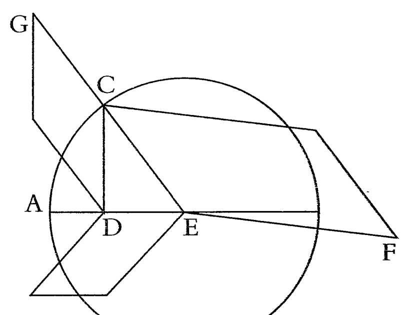
Let the diameter of the circle be straight line AB and its center point E. Let the straight line CD be dropped ordinatewise;

therefore it is perpendicular to AB, since every diameter of a circle is an axis. On the straight lines EA and CD, let the equiangular figures AF and DG be described, and let CD:CG comp AE:EF, the upright:the transverse; therefore, CD:CG::AE:EF, since the upright = the transverse in the case of the circle. Therefore, the two parallelograms AF and DG, having equal angles and proportional sides, are similar. Let a figure be described on ED



similar to the figures on AF and DG. The construction may not look familiar, but if we rearrange the diagram a little, the nature of the proposition will become clear.

Let us now join radius CE and let us reconstruct the parallelogram AE upon this new radius, leaving the parallelogram on



ED in its original location. We shall also draw the parallelogram on CD on the other side of line CD. Applying Apollonius' proof to this case, we shall conclude that figure CF = the figure on DE + the figure on CD.

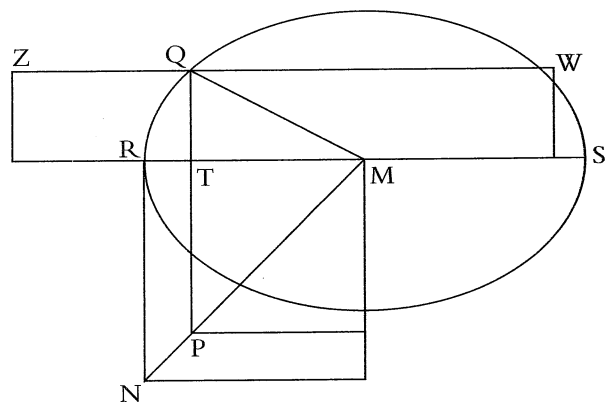
It is now easy to see that our proposition is a case of *Elements* VI, 31: "In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle." In its simplest possible form, where the parallelograms are squares, Apollonius' Proposition 41 is equivalent to *Elements* I, 47, the Pythagorean Theorem! We are justified in seeing the general cases of *Conics* I, 41 as analogues[2] of that famous theorem.

In the case of the ellipse, the figure on the radius AE is equal to the sum of the figures on the ordinate and the line between the ordinate and the center, but now only two of the figures are similar, though all three are equiangular.[3] The bases of the three parallelogrammic figures are not arranged to form a triangle, nor will it always be possible to do so. Three lines may be arranged to form a triangle only if any two taken together are greater than the third (*Elements*, I, 22). Where it is possible, as in the ellipse shown on the next page, the angles of the triangle so formed are of no intrinsic interest. If we have taken the ellipse with its axis, it will always be impossible to form a right triangle from the three lines.

The ratio of the upright to the transverse is determined in a complex manner by the shape of the cone and the orientation of the cutting plane. It is this ratio which, being compounded with the ratio of the sides AE and EF, produces the shape of the figure on the ordinate CD. I will now attempt to show why this makes sense by analyzing a simple case: the ellipse taken with its major axis. In this case, illustrated below, the upright side is smaller than the transverse.

In the circle, the Pythagorean Theorem implies that the square on the ordinate is equal to the difference between the square on the radius and the square on the line from the center to the foot of the ordinate. Let us suppose that the equivalent relation holds for the ellipse. Since line QM (unlike the radius of a circle) has no special significance for the ellipse, we should replace it by a line that does, taking a hint from Apollonius. By calling the semi-diameter the "radius" of the section, he suggests that it is the analogue of the circle's radius.

Next, since we desire to imitate the Pythagorean Theorem as



closely as possible, let us describe squares MN and MP on the semi-major axis and on the line from the center to the ordinate. No absurdity can result from taking two of the three figures to be squares. We must, however, determine the shape of the figure constructed on the ordinate, which by hypothesis is equal to the difference between the two squares.

Now, either the sides of rectangle TW have a comprehensible ratio or they do not. If not, there is nothing further to investigate, but if there is a comprehensible ratio, the simplest possibility will be for the figure to be a rectangle whose shape is determined by the ratio of the transverse to the upright. Since this is the ratio which characterizes the peculiar distortion from circularity of a given ellipse, the assumption that the rectangle's sides have this ratio is plausible. Is there a simple way to verify this guess?

Let us take us take a closer look at the construction. A famil-

iar Euclidean proposition, *Elements* II, 5, comes immediately to mind: Rect RT, RS + sq TM = sq RM.

Clearly, we can construct a figure on the ordinate QT equal to the rectangle RT, TS. Let this be rectangle QT, QZ.

Now consider *Conics* I, 21. This proposition tells us that the square on the ordinate has to the rectangle contained by the segments of the diameter produced by the ordinate the ratio that the upright has to the transverse: sq QT : rect RT, TS :: upright : transverse. *Invertendo*, rect RT, RS : sq QT :: transverse: upright. Substituting, rect QT, QZ : sq QT :: transverse: upright.

Since these are rectangles under a common height, QT, they are in the same ratio as their bases, therefore, QZ : QT :: transverse : upright.

We have now determined the shape of the third figure, given that the other two figures are squares. This fulfills Apollonius' specification for the construction of the figure on the ordinate in the simple case we are considering. If the figures on the radius and its segment are not squares, the shape of the parallelogram on the ordinate will be obtained by compounding the ratio of the sides of the radius's parallelogram with the transverse to upright ratio.[4]

I have shown by the analysis of a simple case how the truth established more generally in Proposition I, 41 could have been discovered. Apollonius' proof, which takes into account diameters other than the axis and the hyperbola as well as the ellipse, is more complicated than the analysis presented above would suggest. Even so, the basis upon which his proof rests is the one revealed in the argument above.

The Pythagorean Theorem and its converse stand at the end of Book I of Euclid's *Elements* and is generally considered to be that to which the entire book is ordered. Although the same cannot be said of Apollonius' Proposition 41, it is nevertheless immediately ordered to the conclusion of Book I, the construction of the sections from their principles in the plane, that is, the diameter or transverse side, the upright side, and angle of the ordinates to the diameter. The comparison of this proposition to

Euclid's makes it both more interesting and more memorable to the student.

NOTES

1 Quotations here and elsewhere are taken from R. Catesby Taliaferro's translation, published in Vol. 11 of the *Great Books of the Western World* (Chicago, 1952).

2 I call this theorem analogous rather than the same because its statement requires principles not necessary for the enunciation of the original theorem. If the circle were essentially a conic section, it would possess the property as a limiting case of the ellipse's property. Since the circle is prior to the conic sections, the reasons why the Pythagorean Theorem holds for it are other than the reasons why it holds for the conics.

3 In the case of the hyperbola, the figure on the radius EA is equal to the difference between the areas on the ordinate and on the line between the center and the ordinate. As is always the case, addition and subtraction are interchanged when one moves from the ellipse to the hyperbola. I shall say no more about the hyperbola. An analogous property also belongs to the parabola (see Prop. 42).

4 Let A=sq on the radius, B=the other square, C=rect on the ordinate=(u/t) A, and let e/f=any ratio. We have proven that A-B=C. Thus, (e/f) A-(e/f)B=(e/f) C=(u/t)(e/f)A. The first two terms represent similar figures, either rectangular or parallelogrammic. The third will be an equiangular parallelogram obtained by compounding the first with the ratio u/t.

THE PHYSICIAN: A NORMATIVE ARTIST
A Brief Analysis

Herbert Ratner, M.D.

NOT all M.D.s are physicians. Some leave the field of medicine entirely. Others remain in medicine in diverse capacities, some proximate, some remote to the work of the physician: medical historians and philosophers, teachers, administrators, researchers and so on. Whereas the above named possessors of M.D. degrees do not need a license to carry on their work, the physician does.

A confusion arises when medicine is defined as both a science and an art for it implies that the physician functions simultaneously as scientist and artist. This is dangerous. It may confuse a patient with a guinea pig. A common belief is that art substitutes for scientific knowledge presently lacking—that the greater the scientific knowledge, the less relevant the art; that ultimately art will not be needed when scientific knowledge is complete. Meanwhile, art is equated with bedside manner, caring, compassion and guesswork which physicians of previous generations were thought to employ as a substitute for knowledge that awaited a later age.[1] But no one who knows anything about the history of medicine should be so arrogant as to believe that present scientific knowledge is free of error and myth. Witness the frequent withdrawal of highly-touted drugs, such as thalidomide and Mer-29, and multiple outbreaks of iatrogenic disease.

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