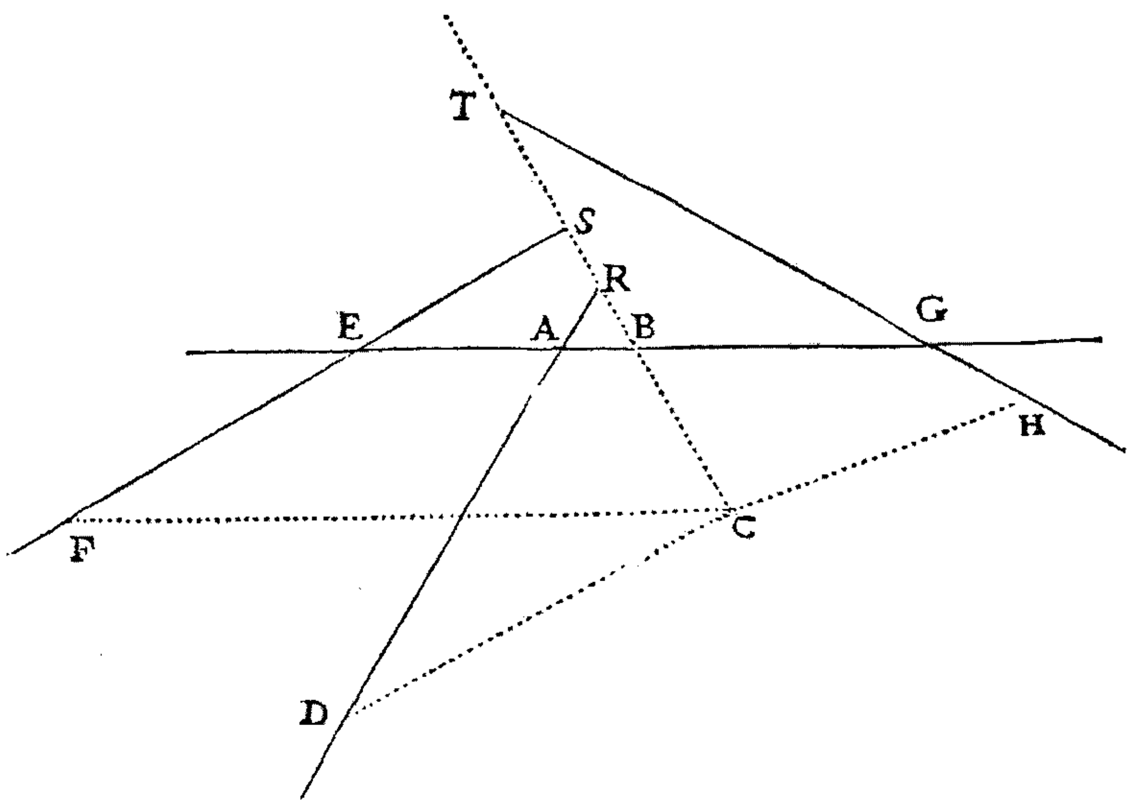
JUNIOR MATHEMATICS READINGS

First Semester



Thomas Aquinas College

[2008-2009 EDITION]

Note on the 2007 edition (Revised 2008)

This edition of the First Semester Junior Math Readings manual was initially motivated merely by the desire to correct the numerous typographical errors in the previous versions. However, when it became clear that most of this manual would have to be scanned into the computer in order to work on it, we realized that this was an opportunity to make a more complete revision. Besides reformatting the text to make it easier to read, the following additional—we hope—improvements have been made.

To the Winfree Smith translation of the *Introduction to the Analytical Art* we have occasionally added, following the pattern of the Smith translation itself, the Latin word or phrase that is being translated. In a very few places, we have also corrected or refined the translation where it may have been misleading or inaccurate, or added a word or two to the footnotes.

Richard Ferrier translated the *Standard Enumeration of Geometrical Results* as part of his master's thesis, and although an anonymous tutor later made another translation of it, we include in this manual the original translation, with only a few revisions suggested by the newer translation (and with reference to the Van Schooten edition). From this original translation we have deleted some of the footnotes, specifically those that pertain to matters not of immediate interest to the students at Thomas Aquinas College, and in a few places we have added the Latin that is being translated.

John Nieto's translation of Descartes's *Geometry* is much more accurate than the D. Smith and M. Latham translation in the Dover edition (1954), but it had a few problems of its own. We have enlarged the font of Dr. Nieto's translation to make it more readable and to allow more margin space for student notes. In a few places the translation itself was corrected. We have also adapted a few of the more helpful footnotes from the Dover edition of the *Geometry*, placing them as footnotes in the running translation. Rather than use our own diagrams, we have inserted facsimiles of the diagrams that appear in the 1637 edition, both to avoid mistakes and to preserve some of the peculiarities that are not easily replicated in most drawing programs (such as the apparently three-dimensional character of the hyper-compass at the beginning of book II). Lastly, we have added a translation (by R. Glen Coughlin) of the section of the *Geometry* on the "normal" for the tutor's discretionary use.

We have also divided Richard Ferrier's *Descartes Notes* into two sections: 1) those specifically explaining part of the text, we have placed as footnotes in the *Geometry* translation; 2) and those that are substantial exercises or problems intended to deepen the student's understanding of Descartes's word we have placed in a section entitled "Exercises and Problems" after the translation of the *Geometry*. The goal here was to diminish the need for the student to be constantly turning pages back and forth between the translation and the notes. We have also not included some of the *Descartes Notes* that are merely corrections of the Dover translation, as these mistakes do not appear in the Nieto translation.

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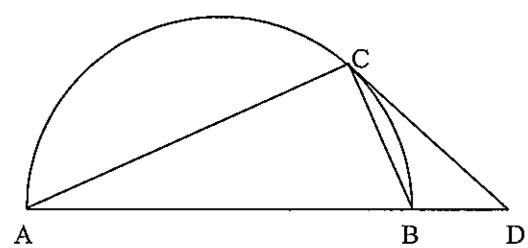
PRELIMINARY NOTE ON ANALYSIS

In this semester, we will be focusing our attention on the work of Viète and Descartes. The title of the first work is *Introduction to the Analytical Art,* and, as we will see, the idea of analysis is central to Descartes's *Geometry.* This idea, however, is not new with them, but is part of the mathematical tradition, extending back even to the Pythagoreans.[1] It will be helpful, therefore, before picking up these works, to have some understanding of this tradition. The following account of analysis is taken from Pappus:[2]

> The so-called Treasury of Analysis, my dear Hermodorus, is, in short, a special body of doctrines furnished for the use of those who, after going through the usual elements, wish to obtain the power of solving theoretical problems which are set to them, and for this purpose only is it useful. It is the work of three men, Euclid the author of the *Elements,* Apollonius of Perga, and Aristaeus the Elder, and proceeds by the method of analysis and synthesis. Now, analysis is the way from what is sought—as if it were admitted—through its concomitants in order to something admitted in synthesis.[3] For in analysis we suppose that which is sought to be already done, and we inquire from what it results, and again what is the antecedent of the latter, until we on our backward way light upon something already known and being first in order. And we call such a method analysis, as being a solution backwards. In synthesis, on the other hand, we suppose that which was reached last in analysis to be already done, and arranging in their natural order as consequents the former antecedents and linking them one with another, we in the end arrive at the construction of the thing sought. And this we call synthesis. Now analysis is of two kinds. One seeks the truth, being called theoretical. The other serves to carry out what was desired to do, and this is called problematical. In the theoretical kind we suppose the thing sought as being and as being true, and then we pass through its concomitants (consequences) in order, as though they were true and existent by hypothesis, to something admitted; then, if that which is admitted be true, the thing sought is true, too, and the proof will be the reverse of analysis. But if we come upon something false to admit, the thing sought will be false, too. In the problematical kind we suppose the desired thing to be known, and then we pass through its consequences in order, as though they were true, up to something admitted. If the thing admitted is possible or can be done, that is, if it is what the mathematicians call given, the desired thing will also be possible. The proof will again be the reverse of analysis. But if we come upon something impossible to admit, the problem will also be impossible.

Pappus gives a multitude of analyses, one of which may serve to illustrate the method *(Collectio* VII, prop. 155).

[To inflect chords into a given circular segment in a given ratio.]



1) Given the circular segment described on AB.
2) To inflect into it the line ACB in a given ratio.
3-5) Let the thing be done, and draw the tangent CD from C; it follows that AD: BD :: sq. AC : sq. BC,

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[1] See T. Heath, *A History of Greek Mathematics*, vol. 1, p. 168.

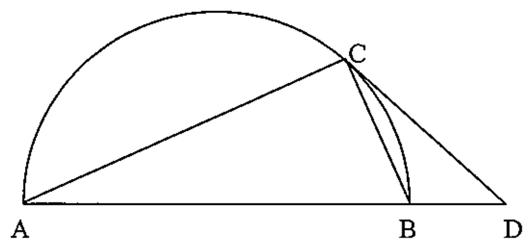
[2] Pappus of Alexandria flourished in the latter part of the third century, A.D. The text quoted is from the beginning of Bk. VII of his *Collectio.*

[3] Analysis and synthesis are transliterations of ανάλυσις [analusis], meaning "taking apart" or "dissolution," and συνθεσις [sunthesis], meaning "putting together" or "composition."

because AD : CD :: CD : BD :: AC : BC.
For since $\angle$ CAD = $\angle$ BCD (*Elements* III, 32), while $\angle$ D is common, triangles CAD & CBD are similar; hence AD : BD dup. AC : BC, or AD : BD :: sq. AC : sq. BC.

6) The ratio of sq. AC : sq. BC is given, so that the ratio AD : AB is given as well.
7) and A and B are given, thus D is given.
8) (and since D is given), C is also given.

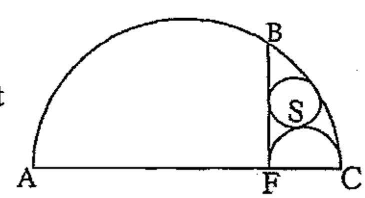
The synthesis of the problem is done in the following way:
9) Let the segment of the circle be ABC and let the given ratio be the ratio of the line E to the line F.
10) Locate D such that AD : DB :: sq. E : sq. F.
11) Draw the tangent DC and join lines AC and CB.
12) I say that the lines AC and CB are the solution to the problem.
13) For since AD : DB :: sq. E : sq. F,
14) And sq. AC : sq. BC :: AD : DB, since DC is tangent,
15) it follows that sq. AC : sq. BC :: sq. E : sq. F.
16) Therefore, AC : BC :: E : F, and the problem is solved.



For additional examples, the reader may wish to refer to the series of propositions beginning with II, 44 in Apollonius' *Conics*.

Problems

1. Produce an analysis of the problem set out by Euclid in *Elements* IV, 10 &11.
2. Using similar figures and the theorem proved above, that the Cartesian products of the means and extremes are equal, reduce the inscription of the regular pentagon to an equation.
3. Inscribe a square in a given right triangle such that the right angle will be in one corner of the square.
4. Inscribe a square in a given right triangle such that the hypotenuse of the triangle contains one side of the square.
5. Inscribe a square in a given triangle.
6. Inscribe a circle in the mixed triangle BFSC in the figure below where ABC, FSC are semicircles, BF1AC:
7. a) Construct a rectangle with a given perimeter, b) Construct a square with a given perimeter.
8. Given line AB cut at C. It is required to cut AB produced beyond B at D
   a) so that rectangle AC,DB shall equal square CD,
   b) so that rectangle AD,BD shall equal square AB.



Before attempting to solve most of these problems, the students should read and discuss Viète's *Introduction to the Analytic Art.*

INTRODUCTION
TO
THE ANALYTICAL ART

(*In Artem Analyticem Isagoge*)

by
François Viète
(Vieta)

published in 1591

Translated by J. Winfree Smith

To the Illustrious Princess Mélusine,
Catherine of Parthenay,
Most Pious Mother of the Lords of Rohan,
I, François Viète of Fontenay,
Pledge Honor and Obedience.[1]

O Princess Mélusine,[2] *most pious mother of the lords of Rohan, the Bretons extol the noble family and ancient ancestry of the house of Rohan, which I do not think could be matched on the whole earth by any other more ancient and illustrious on account of more legitimate possessions or more authentic monuments. They will acknowledge your children as the original stock and as the descendants of the royal blood of Conan, as those who by God's will escaped the yoke of the invader Nominhoë and they will be confident that this noble race will last as long as they, while going about the quarries, woods, and ponds of your domain of Salles, see engraved in marbles, oaks, and scales of fish the insignia of the golden rhomboids*

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[1] This translation is based primarily on the text of the *Isagoge* as republished with annotations in the *Francisci Vietae Opera Mathematica,* ed. F. van Schooten (Leyden, 1646) pp. 1-12, and as much as possible of its style has been preserved. The original edition was also consulted; its full title is: F. Vietae, *In Artem Analyticem* [sic!] *Isagoge.* Seorsim excussa ab *opere restitutae Mathematicae Analyseos, seu* Algebra Nova (*Introduction to the Analytical Art.* Excerpted as a separate piece from the *opus of the restored Mathematical Analysis, or* The New Algebra [Tours, 1591]).

The passages in small italics are the editor's annotation printed in the 1646 edition. The passages in square brackets, as well as the footnotes, have been added by the translator. This translation was made in 1955 at St. John's College in Annapolis.

— J. Winfree Smith.

[2] Catherine of Parthenay (1554-1631) was an ardent Huguenot. After her first husband was killed in the Massacre of St. Bartholomew, she married René of Rohan in Brittany and had by him five children, the eldest of whom, Henri of Rohan, became the famous leader of the Huguenots. Viète had supervised her education and remained her friend and adviser all his life.

Catherine herself was descended from the family of Lusignan, whose ancestral seat was the château of Lusignan, fifteen miles from Poitiers. The legendary ancestress of the family was a fairy named Mélusine. The name was originally Mère des Lusignans, then became Mère Lusigne, afterwards Merlusine, and finally Mélusine. Mélusine had the remarkable ability to turn the lower part of her body into a serpent every Saturday. When she married Raymond, it was on the condition that he would never see her on Saturday. He broke the agreement whereupon she turned completely into a serpent, escaped by the window, and disappeared, only to reappear on the occasion of the death of the lords of Lusignan, when she would utter strange cries of grief. Mélusine was a beneficent fairy and, according to the legend, built the château of Lusignan and many others for her husband.

A Hugh of Lusignan went on the crusade of 1100-1101. Another member of the family, Hugh the Brown, went as a pilgrim to the Holy Land in 1164. In the last quarter of the twelfth century his son Guy became king of Jerusalem and ruler of Cyprus, where his brother's descendants reigned as kings until 1475. In the middle of the fourteenth century, some of the Lusignans of Cyprus went off and made themselves rulers of Armenia, where they held sway from 1342 to 1375. A branch of the family continued in Poitou during the thirteenth century and ruled La Marche until 1303. Hugh of La Marche, whose betrothed wife, Isabel of Angoulême, was seized by King John of England and made his queen, was a nephew of Guy of Lusignan. After John's death Hugh married her and had by her a number of sons who were, therefore, half-brothers of Henry III of England.

The family of René of Rohan owned extensive domains in Brittany, including those of Porhoët and Léon. They were descended from the ancient kings of Brittany the first of whom was Conan Mériadec (409). Judicaël Eudon, and Erech, whom Viète mentions, were all kings of Brittany. St. Mériadec, a descendant of Conan, lived in the seventh century and was bishop of Vannes.

which it wears.[3] *For by their own religious lore (*cabalâ*), the Bretons will testify that as it was granted of His sole favor by God most great and most good to the prayers of St. Mériadec, a former prince of the family, so also now it is granted to me, who marvel at few things, to marvel time and again at the strange warblings of birds and other remarkable things around the sanctuary, which long ago was his, constructed in the midst of woods and pleasant groves. I, of Fontenay in Poitou, a regular inhabitant of the banks of the Vendée, cherish the name* (nomen) *and the majesty* (numen) *of Mélusine and her descendants of the castle constructed long ago by the divine Mélusine, of whom by Raymond you are the blessed progeny. And I also add a prophecy* (omen). *However, I do not for this purpose oppose to the Judicaëls, the Eudons, and the Erechs of the house of Rohan your Guys, Godfreds, Hughs, and Bruns; nor to their Breton kings, princes in Léon, counts in Porhoët, do I oppose your kings of Cyprus, your princes of Antioch and Armenia, your counts of Angoulême and La Marche nor to their Isabel, daughter of the Scot, nor to Isabel of Navarre, do I oppose your Isabel, mother of English kings and of your ancestors of Lusignan. But rather I piously recall and judge that it happened auspiciously and as if by decree of destiny that the goddess Mélusine in gratitude for the help received from René of Rohan, since he had strenuously defended her castle of Lusignan when it was besieged at the instigation of the Guises, forthwith bestowed on him you, her own and Raymond's offspring and heir, and the rule of the family of Rohan. Raymond himself, to be sure, was descended from the family of Rohan, and now the offspring of Raymond and Mélusine were returned to that source from which they first began; thus it will hardly perish, for this circle is a true and truly physical symbol of perpetuity. But even less will your virtues perish in this cyclical restitution of the beginning. And just as our ancestors, in their own idiom, which was then being adopted, called your ancestress "Fairy Mélusine" because of her venerable appearance and her rare and remarkable gifts of mind, so posterity will invoke you as heavenly goddess* (δίαν θεάων) *and will address you as queen* (πότνιαν), *as trustworthy* (κεδνήν) *ruler, and with a more worthy epithet, if any occurs.*[4] *And may the fruits of our nightly labor be pleasing to her, so that she may credit them where they are owed, to you and to your most dear sister Françoise of Rohan, duchess of Nîmes and of Loudinois. For the benefits which you and she bestowed on me in most unhappy times are infinite. How can I adequately commemorate that you delivered me from brigand's chains and from the jaws of death and that, in a word, you helped me with your solicitude and*

[3] A reference to the coat of arms of the Rohan family.

[4] These are all Homeric epithets, applied in Homer to gods and heroes. Cf. *Iliad* XVIII, 388; XIX, 6; *Odyssey* V, 215; XIII, 291; XX, 11; XIV, 170.

generosity as often as my needs and misfortunes prompted you? I owe my life, or if there is anything dearer to me than life, entirely to you; and now, O divine Mélusine, I owe to you especially the whole study of Mathematics, to which I have been spurred on both by your love for it and by the very great skill you have in that art, nay more, the comprehensive knowledge in all sciences (Encyclopaedia) *which can never be sufficiently admired in one of your sex who is of so royal and noble a race. O princess most to be revered, those things which are new are wont in the beginning to be set forth rudely and formlessly and must then be polished and perfected in succeeding centuries. Behold, the art which I present is new, but in truth so old, so spoiled and defiled by the barbarians, that I considered it necessary, in order to introduce an entirely new form into it, to think out and publish a new vocabulary, having gotten rid of all its pseudo-technical terms* (pseudo-categorematis) *lest it should retain its filth and continue to stink in the old way, but since till now ears have been little accustomed to it, it will be hardly avoidable that many will be offended and frightened away at the very threshold. And yet underneath the Algebra or Almucabala which they lauded and called "the great art," all Mathematicians recognized that incomparable gold lay hidden, though they used to find very little. There were those who vowed hecatombs and made sacrifices to the Muses and Apollo if any one would solve some one problem or other of the order of such problems as we solve freely by the score, since our art is the surest finder of all things mathematical.*[5] *Now that the thing has come to pass,* will they be bound by their vows?[6] *However, it would be right for me not now to commend my own wares, but in all moderation yours and those which have been acquired and renewed through your beneficence, to bear witness to my desire that whatever glory is due on account of the felicity of your rule should not be snatched away. For it is not the same in mathematics as in other studies, that everyone's opinion is free, and free his judgment. Here things are done by rule and effort, and neither the persuasions of rhetoricians nor the pleadings of lawyers are of use. The metal which I bring forth yields the kind of gold which they wanted for so long a time. Either that gold is alchemical and faked or it is genuine and true. If it is alchemical, it will surely vanish into smoke, or certainly by the royal touchstone. If on the contrary it is genuine, as it surely is (for I am not one who fights against nature* [φυσιομάχος]*), I yet do not accuse of deceit those who, with every expectation of seeing their work rewarded, enticed others into digging that*

[5] According to legend, Pythagoras sacrificed an ox upon the discovery of the famous Pythagorean theorem. Viète introduces Theorem III of Chapter IX of his *Ad Problema Adriani Romani Responsum* with the words "Moved by the beauty of this discovery, O divine Mélusine, I have sacrificed to you a hundred sheep in place of one Pythagorean ox."

[6] The de-italicization of this phrase is in the original.

gold out of mines hitherto inaccessible and barred by the watchful custody of flame-spouting dragons and other poisonous and deadly serpents, but I fairly ask and expect that they should at least not refuse the support of their authority (which I esteem) against the ignorance or impudence of men who calumniate and detract from another's praise. Therefore, my princess, hold your own work dear and bless it with your blessedness, having referred everything to the supreme ruler of rulers whom you most religiously reverence in soul and in truth (εν ψυχη και αληθεία*), with the praise and glory of all praises. From the marshes of the Isles de Mont of your most dear sister, in the second year of our most Christian and august King* Henry IV,[7] *most zealous and most just punisher of the enemies of the state and the murderers of Christ* (χριστοκτόνων).

[7] See previous note.

Chapter I
On the definition and division of analysis, and those things which are of use to zetetics

In mathematics there is a certain way of seeking the truth, a way which Plato is said first to have discovered,[8] and which was called "analysis" by Theon and was defined by him as "taking the thing sought as granted and proceeding by means of what follows to a truth that is uncontested"; so, on the other hand, "synthesis" is "taking the thing that is granted and proceeding by means of what follows to the conclusion and comprehension of the thing sought."[9] And although the ancients set forth a twofold analysis,[10] the zetetic (ζητητική) and the poristic (ποριστική), to which Theon's definition particularly refers, it is nevertheless fitting that there be established also a third kind, which may be called rhetic or exegetic (ʻρητική η εξεγητική), so that there is a zetetic art by which is found the equation or proportion between the magnitude that is being sought and those that are given, a poristic art by which from the equation or proportion the truth of the theorem set up is investigated (*examinatur*), and an exegetic art by which from the ordered equation or the proportion there is shown (*exhibetur*) the magnitude itself which is being sought. And thus, the whole threefold analytical art, claiming for itself this office, may be defined as the science (*Doctrina*) of right finding in mathematics. Now what truly pertains to the zetetic art is established by the art of logic through syllogisms and enthymemes, the foundations of which are those very stipulations (*symbola*)[11] by which equations and proportions are arrived at, which

[8] See Proclus: *In Euclid,* p. 211, 19-22; and Diogenes Laertius, II, 24. The procedure of a Platonic dialogue may be described as analytical in the sense that it assumes what is being talked about (e.g., "justice") is known and that those who participate in the dialogue may, by talking and reasoning about it as if it were known, attain the knowledge of it. This is the meaning of the myth of recollection.

[9] These definitions of *analysis* and *synthesis* are found in a scholium to Euclid XIII, 1-5 (Heath, *Euclid,* Vol. III, p. 442): "Analysis is the assumption of that which is sought as if it were admitted [and the arrival] by means of its consequences at something admitted to be true. Synthesis is an assumption of that which is admitted [and the arrival] by means of its consequences at something admitted to be true." Most of the extent manuscripts of Euclid are based on an edition of Theon of Alexandria (4th c. A. D.). Hence many contemporaries of Viète, and apparently he also, held that Theon had completely rewritten the original Euclid. Heath claims that the scholium is an interpolation made before Theon's time. An example of what Viète means by analysis would be the solution of a problem in algebra by letting the unknown be *x*, setting up an equation which satisfies the conditions of the problem, and proceeding to the answer. An example of synthesis would be a demonstration.

[10] Pappus (Hultach, II, pg 634, 24-26) distinguishes two kinds of analysis: "τὸ μεν ζητητικὸν τ' αληθους, 'ο καλ– εἰται θεωρητικόν, τὸ δε ποριστικὸν προταθεντος, 'ο καλεἰται προβληματικὸν." In the first of these, the *zetetic*, the "analysis" is concerned with finding the proof of a *theorem* (θεωρημα [theōrēma], "something beheld or contemplated"), and the inversion of the "analysis," i.e., the "synthesis," is the demonstration (αποδειξις [apodeixis]). In the second, the *poristic*, the "analysis" is concerned with the finding of a solution (usually a construction) related to the *problem* (πρόβλημα [problēma], "a shelter, excuse, problem"), and the inversion of the "analysis," or the "synthesis," directly represents a geometrical construction or a porism, which is then followed by a demonstration (αποδειξις). Viète, however, gives quite a different meaning to the distinction between the zetetic and the poristic "analysis." The word *zetetic* is from the Greek verb ζητεω (zeteō) = "to seek," and the word *poristic* is from ποριζω (porizō) = "to provide."

[11] The word *symbolum* has here the somewhat unusual meaning of "something agreed upon." It might be translated "axiom," except that Viète's list of *symbola* includes certain Euclidean theorems as well as axioms. Note also that in

stipulations must be derived from common notions as well as from theorems that are demonstrated by the power of analysis itself. In the zetetic art, however, the form of proceeding is peculiar to the art itself, inasmuch as the zetetic art does not employ its logic on numbers—which was the tediousness of the ancient analysts—but uses its logic through a logistic which in a new way has to do with species.[12] This logistic is much more successful and powerful than the numerical one for comparing magnitudes[13] with one another in equations, once the law of homogeneity has been established; and hence there has been set up for that purpose a series or ladder, hallowed by custom, of magnitudes ascending or descending by their own nature from genus to genus, by which ladder the degrees and genera of magnitudes in equations may be designated and distinguished.

Chapter II
On the stipulations (*symbola*) governing equations and proportions

The Analytical Art assumes as manifest the better known stipulations governing equations and proportions which are to be found in the *Elements*, such as are:[14]

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chapter 4, precepts 1, 2, and 4, *symbolum* is translated as "symbol."

[12] Here the zetetic art, or the art of right finding, is further defined as logistic. It is a "logisticae speciosa," or an art of computing or reckoning with "species." The Latin word "species" has a tremendous history. It translates the Greek word εἴδη [eidē]) which among the Pythagoreans referred to forms and classes of numbers and which with Plato and Aristotle signified whatever in things gives them being and intelligibility. Its use by Viète probably comes ultimately from the Pythagoreans, but through Diophantus. In Diophantus' *Arithmetic* the εἴδη are signs (such as the sign for an unknown number) which serve as tools in solving certain arithmetical problems.

Although this seems the most likely source for Viète's use of the word "species," the Reverend John Wallis in his *A Treatise of Algebra* (London, 1685), p. 66, advances another theory:

> The name of *Specious Arithmetick* is given to it (I presume) with respect to a sense wherein the civilians use the word *Species*; for whereas it is usual with our Common Lawyers to put Cases in the name of John-an-Oaks and John-a-Stiles or John-a-Down, and the like (by which names they mean any person indefinitely who may be so concerned) and of later times (for brevity sake) of J.O. and J.S. or J.D. (or yet more shortly) of A, B, C, etc. In like manner, the Civilians make use of the Names of Titus, Sempronius, Caius, and Mevius or the like, to represent indefinitely, any person in such circumstances. And cases so propounded they call *Species*. Now with respect hereunto, Viète (accustomed to the language of the Civil Law) did give, I suppose, the Name of *Species* to the letters A, B, C, etc., made use of by him to represent indefinitely any Number or Quantity, so circumstanced as the occasion required. And accordingly, the accommodation of Arithmetical Operations to Numbers or other Quantities thus designed by *Symbols* or *Species*, was called *Arithmetica Speciosa* or *Specious Arithmetick*; the word *Species* signifying what we otherwise call *Notes, Marks, Symbols,* or *Characters*, made use of for the compendious expressing or designation of Numbers or other Quantities.

Wallis' theory derives credence from the fact that Viète was a jurist. It may be, of course, that the word "'species" as used by Viète is meant to contain something of the meaning of the Diophantine εἴδη and also something of the juridical meaning.

[13] The appearance of the word "magnitudes" here makes it clear that Viète understands his "species" as identical with the general [?] magnitudes of the 5th Book of Euclid.

[14] Symbolum 2 is the same as Euclid I, Common Notion 1.

1. The whole is equal to its parts.
2. Things which are equal to the same thing are equal to each other.
3. If equals are added to equals, the sums are equal.
4. If equals are subtracted from equals, the remainders are equal.
5. If equals are multiplied by equals, the products are equal.
6. If equals are divided by equals, the results are equal.
7. If any magnitudes are proportional directly, they are proportional inversely and alternately [i.e., if *a:b::c:d*, then *b:a::d:c* and *a:c::b:d*].
8. If like proportionals are added to like proportionals, the sums are proportional [i.e., if *a:b::c:d*, then *a+c:b+d::a:b*].
9. If like proportionals are subtracted from like proportionals, the remainders are proportional [i.e., if *a:b::c:d*, then *a-c:b-d::a:b*].
10. If proportionals are multiplied by proportionals the products are proportional [i.e., if *a:b::c:d* and *e:f::g:h*, then *ae:bf::cg:dh*].

For when proportionals are multiplied by proportionals, the same ratios are being compounded. Now it was commonly received by the ancient geometers that ratios which are compounded of the same ratios are the same with each other, as is seen everywhere in Apollonius, Pappus, and the other geometers. But the compounding of ratios is effected by the multiplication of the antecedents and the consequents, respectively, as is clear from those things that Euclid shows in the twenty-third proposition of the sixth book and the fifth proposition of the eighth book of the Elements.

11. If proportionals are divided by proportionals, the results are proportional [i.e., if *a:b::c:d* and *e:f::g:h*, then *a/e : b/f :: c/g : d/h*].

For when proportionals are divided by proportionals from the same ratios other same ratios are separated, and just as by multiplication ratios are compounded together, so by division one ratio is separated from another; for division undoes what multiplication, as shown, does.

12. The equation or ratio is not changed by a common multiplier or divisor [i.e., *ma:mb::a:b* and *a/m : b/m ::a:b*].
13. Products under the several segments are equal to the product under the whole [i.e., *ab+ac = a(b+c)*].
14. Products obtained by a succession of magnitudes, or quotients obtained by a succession of divisors, are equal, no matter in what order the multiplication or division is done [i.e., *a-b = b-a* and *(a/b)/c = (a/c)/b*].

“	3 “ “ “	“ “	“ “	“	2.
“	4 “ “ “	“ “	“ “	“	3.
“	7 “ “ “	“ “	V, def. 13, 12, and prop. 16.		
“	10 “ “ “	“ “	VI, prop. 23; VIII, prop. 5.		
“	13 “ “ “	“ “	VI, prop. 1.		
Symbola 15 & 16 are the same as VI, prop. 16 and 17; VII, prop. 19.					
Symbolum 5 corresponds to Euclid I, Common Notion 5.					
“	6 “	“ “	“ “	“	6.
“	8 “	“ “	V, prop. 12.		
“	9 “	“ “	“ “	19.	
“	12 “	“ “	VII, “	17.	
“	14 “	“ “	“ “	16.	

But the paramount stipulation governing equations and proportions and the one that is all-important in analysis is:

15. If there be three or four magnitudes and the result of the multiplication of the extreme terms is equal to the result of the multiplication of the mean by itself or to the product of the means, then those magnitudes are proportional [i.e., if $ab = cd$, then $a:c::d:b$; or if $ab = c^2$ then $a : c :: c : b$]. And conversely.
16. If there be three or four magnitudes, and as the first is to the second, so that second, or else some third, is to another, the product of the extreme terms will be equal to the product of the means [i.e., if $a : b :: c : d$, then $ad = bc$; and if $a : b :: b : c$, then $ac= b^2$].
17. And so, a proportion can be called the composition (*constitutio*) of an equation, an equation the resolution (*resolutio*) of a proportion.

Chapter III
Concerning the law of homogeneity and the degrees and genera of the magnitudes that are compared (*comparatarum*)[15]

The supreme and everlasting law of equations or proportions, which is called the law of homogeneity because it is conceived with respect to homogeneous magnitudes, is this:

1. Only homogeneous magnitudes are to be compared (*comparari*) with one another.

For, as Adrastus[16] said, it is impossible to know how heterogeneous magnitudes may be conjoined.

And so, if a magnitude is added to a magnitude, it is homogeneous with it.

If a magnitude is multiplied by a magnitude, the product is heterogeneous in relation to both.

If a magnitude is divided by a magnitude, it is heterogeneous in relation to it.

Not to have considered these things was the cause of the darkness and blindness of the

[15] Comparison (*comparatio*) means, on the one hand, adding and subtracting magnitudes to form algebraic expressions and. on the other, equating magnitudes or expressions with one another. Cf. Descartes, *Rules for the Direction of the Mind*, Rule XIV.

[16] Theon (Miller), p. 73,18f: "For Adrastus says that it is impossible to know how heterogeneous magnitudes may be in a ratio to one another." Adrastus' remark is a comment on Euclid V, Def. 3: "A ratio is a sort of relation with respect to size between magnitudes of the same kind." The meaning of this definition is that we cannot, for example, conceive of a ratio of a rectangle to a straight line, since these are magnitudes of different kinds. Viète alters the meaning of Adrastus' statement, taking it to mean only that magnitudes cannot be added or subtracted from one another unless they are magnitudes of the same kind. Viète would say that x^2 and x are not magnitudes of the same kind and that, therefore, we cannot write $x^2 + x$ or $x^2 - x$, as we do today. He thinks of the "species", which are general magnitudes and not lines or surfaces or solids or weights or times, as having a quasi-geometrical character. That is to say, the unit in relation to which x^2 is determined is different in kind from the unit in relation to which x is determined. Descartes shows (*Geometry*, bk. 1) that the ordinary operations of arithmetic can be defined in such a way that the results, including the sums, differences, products, quotients, roots, or powers of given magnitudes, can be understood as being of the same dimension as the original magnitudes (cf. Rule XIV of the *Rules for the Direction of the Mind*). We almost instinctively think of x^2 and x as numbers, and the unit of all numbers is the same. Viète thought of them as being like numbers, for they can be multiplied, divided, etc., but they are not the same as numbers.

ancient analysts.

2. Magnitudes which by their own nature (*sua vi*) ascend and descend proportionally from genus to genus may be called "ladder rungs."[17]

3. The first of the ladder magnitudes is "side" (*latus*) or "root" (*radix*).
The second is "square" (*quadratum*).
The third is "cube" (*cubus*).
The fourth is "squared-square" (*quadrato-quadratum*).
The fifth is "squared-cube" (*quadrato-cubus*).
The sixth is "cubed-cube" (*cubo-cubus*).
The seventh is "squared-squared-cube" (*quadrato-quadrato-cubus*).
The eighth is "squared-cubed-cube" (*quadrato-cubo-cubus*).
The ninth is "cubed-cubed-cube" (*cubo-cubo-cubus*).
And those remaining may be denominated from these by this series and method.

4. The genera of the compared (*comparatarum*) magnitudes, so that they may be equated in an orderly way to the ladder magnitudes, are:[18]
The first, "length" (*longitudo*) or "breadth" (*latitudo*),
The second, "plane" (*planum*),
The third, "solid" (*solidum*),
The fourth, "plane-plane" (*plano-planum*),
The fifth, "plane-solid" (*plano-solidum*),
The sixth, "solid-solid" (*solido-solidum*),
The seventh, "plane-plane-solid" (*plano-plano-solidum*),
The eighth, "plane-solid-solid" (*plano-solido-solidum*),
The ninth, "solid-solid-solid" (*solido-solido-solidum*).
And the remaining ones may be denominated from these by this series and method.

5. Of ladder magnitudes, the higher degree in relation to the "side" (*latus*) as the lowest and that to which the compared magnitude corresponds, is called the "power" (*potestas*). The other, lower, ladder magnitudes are called degrees "on the way (*parodici*) to the power" [translated simply by "lower"].

6. The power is pure when it is free from "conjoined" magnitudes. If the power is joined with a magnitude which is the product of a lower rung and a coefficient, it is a "conjoined" power [x^5 is a pure power; x^5+ax^4 is a "conjoined" power].

Pure powers are: "square," "cube," "squared-square," "squared-cube," "cubed-cube," etc.
Conjoined powers are:
At the second rung: a "square" together with a "plane" which is the product of a "side" and a "length" or "breadth" [x^2+ax];
At the third rung:

(i) a "cube" together with a "solid" which is the product of a "square "and a "length" or "breadth" [x^3+ax^2],

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[17] The "ladder-rungs" are successive powers of a single unknown, which powers are in proportion. Thus, in modem notation, $x{:}x^2{::}x^2{:}x^3{::}x^3{:}x^4$, etc. The "ladder-rungs" mean, then, different degrees of the unknown.

[18] The equated magnitudes would be not simply known magnitudes which we nowadays would designate by such letters as *a*, *b*, or *c* and which by the law of homogeneity would have to be understood as "lengths" or "planes" or "solids" according as they are equated with x or x^2 or x^3, but also, as appears in the sequel, products of known and unknown magnitudes. Thus, ax^2 would be a product of a "length" and a "square." It would itself be a "solid" and might be equated with x^3.

(ii) *a "cube" together with a "solid" which is the product of a "side" and a "plane" [x^3+bx, where b is understood as a "plane " magnitude],*

(iii) *a "cube" together with a "solid" which may be either the product of a "square" and a "length" or "breadth" or the product of a "side" and a "plane" [x^3+c, where c is understood as a "solid" magnitude; c can equal mx^2 or dx, where d is a "plane" magnitude];*

At the fourth rung:

(i) *a "squared-square" together with a "plane-plane" which is the product of a "cube" and a "length" or "breadth" [x^4+ax^3],*

(ii) *a "squared-square" together with a "plane-plane" which is the product of a "square" and a "plane" [x^4+bx^2, where b is a "plane"],*

(iii) *a "squared-square" together with a "plane-plane" which is the product of a "side" and a "solid" [x^4+cx, where c is a "solid"],*

(iv) *a "squared-square" together with a "plane-plane" which is either the product of a "cube" and a "length" or "breadth" or the product of a "square" and a "plane" [x^4+c, where c is a "plane-plane"; c can be equal to mx^3, or to dx^2, where d is a "plane"],*

(v) *a "squared-square" together with a "plane-plane" which is either the product of a "cube" and a "length" or "breadth" or the product of a "side" and a "solid" [x^4+c, where c is a "plane-plane"; c can equal mx^3 or dx where d is a "solid"],*

(vi) *a "squared-square" together with a "plane-plane" which is either the product of a "square" and a "plane" or of a "side" and a "solid" [x^4+c, where c is a "plane-plane"; c can equal mx^2 where m is a "plane" or dx where d is a "solid"],*

(vii) *a "squared-square" together with a "plane-plane" which is either the product of a "cube" and a "length" or "breadth" or of a "square" and a "plane" or of a "side" and a "solid" [x^4+c, where c is a "plane-plane"; c can equal mx^3 or dx^2, where d is a "plane," or ex, where e is a "solid"].*

In the same order the conjoined powers at the remaining rungs of the ladder may be found. But if we want to know how many genera of conjoined powers are at each rung, let there be taken a number less by unity than that term which is produced by geometric progression from unity in the double ratio [1:2::2:4::4:8, etc.] and which has the same ordinal position as the power under consideration. Thus, if one wants to know how many conjoined powers are at the rung of the "squared-square", i.e., at the fourth rung, the fourth term of the geometric progression, namely 8, must be taken, from which, when the unit has been taken away, 7 remains. And so, there are at the fourth rung as many conjoined powers as we have just enumerated. By this procedure it will be found that at the rung of the "squared-cube," i.e., the fifth rung, there are fifteen genera of conjoined powers.

7. Coefficient[19] magnitudes which multiply ladder magnitudes that are lower in relation to a certain power and thus produce a homogeneous magnitude to be added to that power shall be called "subrungs."

The "subrungs" are "lengths" or "breadths," "plane," "solid," "plane-plane," etc. Thus, if there be a "squared-square" to which there is joined a "plane-plane" which is the product of a "side" and a "solid," the "solid" magnitude will be the "subrung"; and in relation to the "squared-square" the "side" will be a lower ladder magnitude. [In the expression x^4+cx it is apparent that x^4 is a "squared-square"; cx is a "plane-plane," being the product of the "side" x and the "solid" c; c, then, is the subrung, and x is in relation to x^4 a lower ladder magnitude.] Or if there be a "squared-square" together with a "plane-plane," which is either the product of a "square" and a "plane" or the product of a "side" and a "solid," the "plane" and the "solid" will be "subrung" magnitudes; and in relation to the "squared-square" the "square" and the "side" will be lower ladder magnitudes. [Thus we may have x^4+cx, as above, or x^4+cx^2, where cx^2 is a "plane-plane" and c is understood as a "plane." Then the "plane" c is the "subrung," and the "square" x^2 is a lower ladder magnitude in relation to x^4.]

[19] The term "coefficient" was introduced by Viète into the terminology of algebra.

Chapter IV
On the precepts of the reckoning by species

The numeral reckoning (*logistice numerosa*) operates with numbers; the reckoning by species (*logistice speciosa*) operates with species or forms of things,[20] as, for example, with the letters of the alphabet.

Diophantus has handled the numerical reckoning in the thirteen books of the Arithmetic, *of which only the first six are extant, but which are now available in Greek and Latin and elucidated by the very learned commentaries*[21] *of a most illustrious man, Claude Bachet [de Méziriac]. But Viète has produced the reckoning by species in the five books of the* Zetetics, *which he has arranged chiefly from selected problems of Diophantus, some of which he solves by a method peculiar to himself. Wherefore, if you wish to discern profitably the distinction between the two kinds of reckoning, you must consult Diophantus and Viète together, and the zetetics of the latter must by viewed along with the arithmetical problems of the former; it is in order that I may lighten for you the labor of this task that I shall briefly note the zetetics which have been taken from the problems of Diophantus* (see following table).

Diophantus		Viete	
Book of the Arithmetic	*Problem*	*Book of the Zetetics*	*Problem*
I	1	I	1
	4		2
	2		3
	7		4
	9		5
	5		7
	6		8
II	8, 9	IV	1
	10		2, 3
	11		6
	12		7
	13		8
	14		9
	8		11
III	7, 8	V	1
	9		3
	10		4
	11		5
	12		7
	13		8
V	9		9
IV	34		13

There are four canonical precepts for reckonings by species (*logistices speciosae*).

[20] It is not quite clear what Viète means when he calls the "species" forms of things (*formae rerum*). The best guess is that we have here a kind of ghost of the Pythagorean and Platonic εἴδη.

[21] First printed in 1621.

Precept I

To add a magnitude to a magnitude

Let there be two magnitudes *A* and *B*. It is required (*oportet*) to add the one to the other.

Since, therefore, a magnitude is to be added to a magnitude, but heterogeneous magnitudes cannot be conjoined, those which are proposed to be added to one another are two homogeneous magnitudes. However, that one of them is greater or less than the other does not imply that they are of different genera. Therefore, they may be fittingly added[22] by [means of] the sign for coupling or addition (*nota copulae seu additionae*); and, put together, they will be *A* "plus" *B*, if they are simple "lengths" or "breadths."

But if they stand higher on the aforesaid ladder or if they share a genus with those that stand higher, they will be designated by the appropriate denominations, as, for instance, we may say, "*A* square 'plus' *B* plane" or "*A* cube 'plus' *B* solid," and similarly in other cases.

The analysts, however, are accustomed to indicate the performance of addition by the symbol +.[23]

Precept II

To subtract (subducere) *a magnitude from a magnitude*

Let there be two magnitudes *A* and *B*, and let the former be greater than the latter. It is required to subtract the less from the greater.

Since, then, a magnitude is to be subtracted from a magnitude, but heterogeneous magnitudes cannot be conjoined, those which are proposed are two homogeneous magnitudes. That one of them is greater and the other less does not imply that they are of different genera. Therefore, subtraction may be fittingly effected (*fiet*) by [means of] the sign of the disjoining or removal[24] of the less from the greater; and disjoined, they will be *A* "minus" *B*, if they are simple "lengths" or "breadths."

But if they stand higher on the aforesaid ladder or if they share a genus with those that stand higher, they will be designated by the appropriate denominations, as, for example, we may say "*A* square 'minus' *B* plane" or "*A* cube 'minus' *B* solid," and similarly in the other cases.

Nor will it be done differently if the magnitude which is subtracted is itself conjoined with some magnitude, since the whole and the parts are not to be judged by separate laws; thus, if "*B* 'plus' *D*" is to be subtracted from *A*, the remainder will be "*A* 'minus' *B*, 'minus' *D*," the magnitudes *B* and *D* having been subtracted one by one.

But if *D* is already subtracted from *B* and "*B* 'minus' *D*" is to be subtracted from *A*, the result will be "*A* 'minus' *B* 'plus' *D*," because in the subtraction of the whole magnitude *B* that which is subtracted exceeds by the magnitude *D* what was to have been subtracted. Therefore, it must be made up by the addition of that magnitude *D*.

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[22] For the first time in history, to the signs as signs a numerical property is attributed. *A* and *B* are added *as if* they were particular numbers or magnitudes.

[23] This symbol is an abbreviation of the Latin "*et*". Together with the minus symbol (-), it first came into use in Germany toward the end of the fifteenth century.

[24] "Removal" here translates a juridical term "*multa*," which means a fine, and is preserved in the English word "mulct.'

The analysts, however, are accustomed to indicate the performance of the removal by means of the symbol –. And this is "defect" (λείψις) in Diophantus, as the performance of addition is "presence" (ʽυπαρξις).

But when it is not said which magnitude is greater or less, and yet the subtraction must be made, the sign of the difference is: =, i.e., when the less is undetermined (*incerto*); as, if "*A* square" and "*B* plane" are the proposed magnitudes, the difference will be: "*A* square = *B* plane," or "*B* plane = *A* square."[25]

Precept III

To multiply (ducere) *a magnitude by a magnitude* (in magnitudinem)

Let there be two magnitudes *A* and *B*. It is required to multiply the one by the other.

Since, then, a magnitude is to be multiplied by a magnitude, they will by their multiplication produce a magnitude heterogeneous in relation to each of them; and therefore, their product (*quae sub iis*) will rightly be designated by the word "*in*" or "*sub*," as, for example, "*A* in *B*," by which it will be signified that the one has been multiplied by the other; or as others say, that a magnitude is produced (*facta*) "under" *A* and *B*, and that simply, if *A* and *B* are simple "lengths" or "breadths."[26]

But if they stand higher on the ladder or if they share in genus with magnitudes that stand higher, it is agreed to add the names themselves of the ladder magnitudes or of those that share in their genus, as, for example, "*A* square in *B*" or "*A* square in *B* plane" or "*A* square in *B* solid," and similarly in the other cases.

If, however, the magnitudes to be multiplied, or one of them, be of two or more names, nothing different happens in the operation.[27] Since the whole is equal to its parts, therefore also the products (*facta*) under the segments of some magnitude are equal to the product under the whole. And when the positive name[28] (*nomen adfirmatum*) of a magnitude is multiplied by a name also positive of another magnitude, the product will be positive, and when it is multiplied by a negative name (*nomen negatum*), the product will be negative.

From which precept it also follows that by the multiplication of negative names by each other a positive product is produced, as when "$A - B$" is multiplied by "$D - G$" [giving $DA - DB - GA + GB$], since the product of the positive *A* and the negative *G* is negative, which means that too much is removed or taken away, inasmuch as *A* is, inaccurately, brought forward (*producta*) as a magnitude to be multiplied [as a whole, i.e., in the factor ($-AG$)], and since, similarly, the product of the negative *B* and the positive *D* is negative, which again means that too much is removed, inasmuch as *D* is, inaccurately, brought forward as a magnitude to be multiplied [i.e.,

[25] The introduction of negative quantities makes it no longer necessary to distinguish the two minus signs. The second of these signs, which is now used to signify equality, was so used as early as 1557 by Robert Recorde in his *The Whetstone of Witte*. Viète has no symbol for equality.

[26] These two ways of describing multiplication rise from two different terminologies, the one arithmetical, the other geometrical. In the first case, one magnitude is said to be multiplied into (*ducta in*) the other, while in the second case, the product is said to be produced under (*facta sub*) the factors, as a rectangle is described as being "under" its sides.

[27] That is, $a(u+v)+a(x+y)=au+av+ax+ay$, or $a(x+y+z)=ax+ay+az$ (cf. Euclid II, 1).

[28] The "names," i.e., the signs themselves, are multiplied together *as if* they were particular numbers.

in (–*DE*)]. Therefore, by way of compensation, when the negative *B* is multiplied by the negative *G*, the product is positive.

The denominations of products made by magnitudes ascending proportionally from genus to genus are related to one another in precisely the following way:

A "side" multiplied by itself produces a "square."
A "side" multiplied by a "square" produces a "cube."
A "side" multiplied by a "cube" produces a "squared-square."
A "side" multiplied by a "squared-square" produces "squared-cube."
And interchangeably, i.e., a "square" multiplied by a "side" produces a "cube," a "cube" multiplied by a "side" produces a "squared-square," etc.
Again,
A "square" multiplied by itself produces a "squared-square."
A "square" multiplied by a "cube" produces a "squared-cube."
A "square" multiplied by a "squared-square" produces a "cubed-cube."
And interchangeably.
Again,
A "cube" multiplied by itself produces a "cubed-cube."
A "cube" multiplied by a "squared-square" produces a "squared-squared-cube."
A "cube" multiplied by a "squared-cube" produces a "squared-cubed-cube."
A "cube" multiplied by a "cubed-cube" produces a "cubed-cubed-cube."
And interchangeably, and so on in that order.
In like manner, among the homogeneous magnitudes,
A "breadth" multiplied by a "length" produces a "plane."
A "breadth" multiplied by a "plane" produces a "solid."
A "breadth" multiplied by a "solid" produces a "plane-plane."
A "breadth" multiplied by a "plane-plane" produces a "plane-solid."
A "breadth" multiplied by a "plane-solid" produces a "solid-solid."
And interchangeably.
A "plane" multiplied by a "plane" produces a "plane-plane."
A "plane" multiplied by a "solid" produces a "plane-solid."
A "plane" multiplied by a "plane-plane" produces a "solid-solid."
And interchangeably.
A "solid" multiplied by a "solid" produces a "solid-solid."
A "solid" multiplied by a "plane-plane" produces a "plane-plane-solid."
A "solid" multiplied by a "plane-solid" produces a "plane-solid-solid."
A "solid" multiplied by a "solid-solid" produces a "solid-solid-solid."
And interchangeably, and so on in that order.

Precept IV
To divide (adplicare) *a magnitude by a magnitude*

Let there be two magnitudes *A* and *B*. It is required to divide the one by the other.

Since, then, a magnitude is to be divided by a magnitude, namely higher ones by lower ones, i.e., magnitudes of one kind by magnitudes of another kind, the proposed magnitudes are different in kind. Let *A*, if you will, be a "length" and *B* "plane." And then let a horizontal line appropriately stand between the higher magnitude *B* which is being divided and the lower *A* by which the division is made.

But the magnitudes themselves, i.e., the resultant quotients, will be denominated in accordance with their own rungs at which they are fixed or to which they have been reduced in the ladder of proportional or homogeneous magnitudes, as, for example: $\frac{B\ plane}{A}$, by which symbol the "length" which results from the division of "*B* plane" by "*A* length" may be signified.

And if *B* is given as a "cube" and *A* as a "plane," the result will be $\frac{B\ cube}{A\ plane}$, by which symbol the "length" which results from the division of "*B* cube" by "*A* plane" may be signified.

And if *B* is assumed to be a "cube" and *A* a "length," the result will be $\frac{B\ cube}{A}$, by which symbol the "plane" which arises from the division of "*B* cube" by *A* may be signified, and so on in that order, *in infinitum*.

Nor will anything different be observed among binomial or polynomial magnitudes.

The denominations of the magnitudes that arise from dividing by magnitudes that ascend proportionally by degrees from genus to genus are related to one another in precisely the following way:

A "square" divided by a "side" gives a "side."
A "cube" divided by a "side" gives a "square."
A "squared-square" divided by a "side" gives a "cube."
A "squared-cube" divided by a "side" gives a "squared-square."
A "cubed-cube" divided by a "side" gives a "squared-cube."
And interchangeably, i.e., a "cube" divided by a "square" gives a "side", a "squared-square" divided by a "cube" gives a "side," etc.
Again,
A "squared-square" divided by a "square" gives a "square."
A "squared-cube" divided by a "square" gives a "cube."
A "cubed-cube" divided by a "square" gives a "squared-square."
And interchangeably.
Again,
A "cubed-cube" divided by a "cube" gives a "squared-square" [*sic*!; should be "cube"].
A "squared-cubed-cube" divided by a "cube" gives a "squared-cube."
A "cubed-cubed-cube" divided by a "cube" gives a "cubed-cube."
And interchangeably, and so on in that order.
In like manner, among the homogeneous magnitudes, a "plane" divided by a "breadth" gives a "length."
A "solid" divided by a "breadth" gives a "plane."
A "plane-plane" divided by a "breadth" gives a "solid."
A "plane-solid" divided by a "breadth" gives a "plane-plane."
A "solid-solid" divided by a "breadth" gives a "plane-solid."
And interchangeably.
A "plane-plane" divided by a "plane" gives a "plane."
A "plane-solid" divided by a "plane" gives a "solid."
A "solid-solid" divided by a "plane" gives a "plane-plane."

And interchangeably.
A "solid-solid" divided by a "solid" gives a "solid."
A "plane-plane-solid" divided by a "solid" gives a "plane-plane."
A "plane-solid-solid" divided by a "solid" gives a "plane-solid."
A "solid-solid-solid" divided by a "solid" gives a "solid-solid."
And interchangeably, and so on in that order.

Moreover, if the magnitude that is being divided be the sum, difference, product, or quotient of other magnitudes, nothing prevents the aforesaid precepts from applying to the division, it being noted that, when the magnitude that is being divided, whatever may be its rung, is the product of some magnitude and a magnitude that is the same as the divisor, nothing either in genus or value is added to or taken away from the factor that is not the same as the divisor and that also arises from the division, since what multiplication does division undoes: for example, $\frac{B\ in\ A}{B}$ is A, and $\frac{B\ in\ A\ plane}{B}$ is "A plane."

And thus, in the case of additions, let it be required to add Z to $\frac{A\ plane}{B}$. The sum will be

$$\frac{(A \text{ plane})+(Z \text{ in } B)}{B}$$

[i.e., $a^2/b + z = (a^2 + zb)/b$].

Or let it be required to add $\frac{Z\ square}{G}$ to $\frac{A\ plane}{B}$. The sum will be

$$\frac{(G \text{ in } A \text{ plane}) + (B \text{ in } Z \text{ square})}{B \text{ in } G}$$

In the case of subtractions, let it be required to subtract Z from $\frac{A\ plane}{B}$. The remainder will be

$$\frac{(A \text{ plane}) - (Z \text{ in } B)}{B}$$

Or, let it be required to subtract $\frac{Z\ square}{G}$ from $\frac{A\ plane}{B}$. The remainder will be

$$\frac{(A \text{ plane in } G) - (Z \text{ square in } B)}{B \text{ in } G}$$

In the case of multiplications, let it be required to multiply $\frac{A\ plane}{B}$ by B. The result will be A plane.

Or let it be required to multiply $\frac{A\ plane}{B}$ by Z. The result will be $\frac{A\ plane\ in\ Z}{B}$.

Or, finally, let it be required to multiply $\frac{A\ plane}{B}$ by $\frac{Z\ square}{G}$. The result will be $\frac{A\ plane}{B\ in\ G} in\ Z\ square$.

In the case of division, let it be required to divide $\frac{A\ cube}{B}$ by D. Each magnitude having been multiplied by B, the result will be $\frac{A\ cube}{B\ in\ D}$ [i.e., $(a^3/b)/d = a^3/bd$].

Or let it be required to divide B in G by $\frac{A\ plane}{D}$. Each magnitude having been multiplied by D, the result will be $\frac{B\ in\ G\ in\ D}{A\ plane}$.

Or, finally, let it be required to divide $\frac{B\ cube}{Z}$ by $\frac{A\ cube}{D\ plane}$. The result will be $\frac{B\ cube\ in\ D\ plane}{Z\ in\ A\ cube}$.

Chapter V
Concerning the laws of zetetics

The way to do zetetics is, in general, encompassed in the following laws:

1. If it is a "length" that is being sought, but the equation or proportion is hidden under the wrappings[29] of what is given in the problem, let the unknown which is being sought be a "side."

2. If it is a "plane" that is being sought, but the equation or proportion is hidden under the wrappings of what is given in the problem, let the unknown which is being sought be a "square."

3. If it is a "solid" that is being sought, but the equation or proportion is hidden under the wrappings of what is given in the problem, let the unknown which is being sought be a "cube."

Accordingly, that magnitude which is being sought will by its own nature ascend or descend through the several rungs of the magnitudes that are compared or equated with it

4. Let the magnitudes that are given, as well as those that are being sought, be assimilated and compared (in accordance with the condition dictated by the problem) by adding, subtracting, multiplying, and dividing, the constant law of homogeneity being everywhere observed.

Accordingly, it is clear that finally something will be found which is equal to the magnitude that is being sought or to the power to which it ascends, and that that will consist either entirely of given magnitudes or partly of given magnitudes and partly of the unknown which is being sought or of magnitudes lower than it on the ladder.[30]

5. In order that this work may be assisted by some art, let the given magnitudes be

[29] In solving a problem by algebra the equation may not emerge immediately from the given conditions. It may take some reflection before one sees the equation that satisfies the conditions. This is what Viète means when he speaks of the equation as "hidden under the wrappings of what is given in the problem" (cf. Descartes, *Geometry* [Dover, 1954], pp. 6-8).

[30] In the equation $x^2 = ab$, x^2 is the magnitude which is being sought, and ab is equal to it, and is a product entirely of given magnitudes. In the equation $x^3 = ax^2$, x^3 is the unknown which is being sought, and ax^2 is equal to it and is a product partly of the given magnitude a and partly of a magnitude lower than x^3 on the ladder, namely x^2.

distinguished from the undetermined unknowns (*incertis quaesititiis*) by a constant, everlasting and very clear symbol (*symbolo*), as, for instance, by designating the unknown magnitude by means of the letter *A*, or some other vowel *E, I, O, U*, or *Y*, and the given magnitudes by means of the letters *B, G*, and *D* or the other consonants.[31]

6. Products composed entirely of given magnitudes may be added to one another, or subtracted from one another, according to the sign of their conjunction, and may merge into one product, which shall be the homogeneous element of the equation (*comparationis*), i.e., the element under a given measure; and it shall constitute one side of the equation.[32]

7. In like manner, products composed of given magnitudes and of the same lower ladder magnitude may be added one to another, or subtracted one from another, according to the sign of their conjunction, and may merge into one product which shall be the element homogeneous in conjunction, or the element under the rung [of the lower ladder magnitude].[33]

8. Elements which are homogeneous under the rungs of lower ladder magnitudes shall accompany the power with which they are conjoined, and, along with that power, shall constitute one side of the equation. And thus, the element that is homogeneous under a given measure will be equated to a power designated in its own genus or order; simply, if that power is free from all conjunction with other magnitudes, but if magnitudes homogeneous in conjunction accompany it, which magnitudes are indicated both by the symbol of the conjunction and by the rung of the lower ladder magnitudes, then the magnitude homogeneous under a given measure will be equated not only to it, but to it along with the magnitudes that are products of rungs and coefficient magnitudes.[34]

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[31] This, of course, differs from the convention of present-day algebra, according to which the letters at the end of the alphabet (*x, y, z*, ...) are used to represent unknowns and the letters at the beginning (*a, b, c*, ...) represent knowns. Thomas Harriot in his *Artis analyticae praxis* (1631) followed Viète in using vowels for unknowns and consonants for known quantities, except that he substituted small letters for Viète's capitals. Descartes in his *Geometry* (1637) introduced the system we use now.

[32] In Viète's symbols, "*B* in *C*" and "*D* in *F*" would be products composed entirely of given magnitudes, which products may be added to or subtracted from one another by means of the plus sign or the minus sign. When so added or subtracted, they become "*B* in *C* + *D* in *F*" or "*B* in *C* – *D* in *F*." If we were then to form the equation "*A* square is equal to *B* in *C* + *D* in *F*," then "*B* in *C* + *D* in *F*" would be homogeneous with "*A* square." Since "*B* in *C*" and "*D* in *F*" belong to the rank of "planes," the unit measure is given as a "plane" unit.

In modern notation this would be $x^2 = ab + cd$.

[33] For example, "*A* square in *B*" and "*A* square in *C*" would be products composed of the given magnitudes *B* and *C* and of the same lower ladder magnitude "*A* square." They may be conjoined by means of the plus sign or the minus sign, and then we get either "*A* square in *B* + *A* square in *C*" or "*A* square in *B* – *A* square in *C*." Each is a product under the lower ladder magnitude "*A* square," which is lower in relation to the rung of the product. "*A* square in *B* + *A* square in *C*" or "*A* square in *B* – *A* square in *C*" is called the element homogeneous in conjunction because it is regarded as something to be "conjoined" to a pure power "*A* cube" with which it is homogeneous.

In modern notation this would be $ax^2 + bx^2$ or $ax^2 - bx^2$, either of which binomials would be the element homogeneous in conjunction because it would be considered as "conjoined" with x^3 to make $x^3 + ax^2 + bx^2$, $x^3 + ax^2 - bx^2$, etc.

[34] "*A* square in *G*" and "*A* square in *H*" are elements homogeneous under the rung "*A* square." They accompany the power "*A* cube," are conjoined with it by addition or subtraction, and with it constitute one side of the equation. "*B* plane in *C* + *D* solid" is an element homogeneous under a given measure. It might be equated to "*A* cube" simply, in which case we would have the equation: "*A* cube is equal to *B* plane in *C* + *D* solid"; or "*A* cube" might be accompanied by "*A* square in *G* – *A* square in *H*," a magnitude homogeneous in conjunction, in which case we might have this equation: "*A* cube + *A* square in *G* – *A* square in *H* is equal to *B* plane in *C* + *D* solid."

In modern notation the last equation would be $x^3 + ax^2 - bx^2 = cd + e$, where *c* is understood as a "plane," *d*

9. And, therefore, if the element that is homogeneous under a given measure happens to be mingled with the element that is homogeneous in conjunction, there shall be antithesis.[35]

There is antithesis when positively or negatively conjoined magnitudes cross from one side of the equation to the other under the opposite signs of conjunction, by which operation the equation is not changed. But that must now be demonstrated.

Proposition I

An equation is not changed by antithesis

Let it be given that "*A* square 'minus' *D* plane" is equal to "*G* square 'minus' *B* in *A*."

I say that "*A* square 'plus' *B* in *A*" is equal to "*G* square 'plus' *D* plane," and that by this transposition under opposite signs of conjunction the equation is not changed. For since "*A* square 'minus' *D* plane" is equal to "*G* square 'minus' *B* in *A*," let there be added to both sides "*D* plane 'plus' *B* in *A*." Therefore, from the common notion, "*A* square 'minus' *D* plane 'plus' *D* plane 'plus' *B* in *A*" is equal to "*G* square 'minus' *B* in *A* 'plus' *D* plane 'plus' *B* in *A*." Now the negative conjunction on the same side of an equation cancels the positive. On the one side, the conjunction of "*D* plane" vanishes; on the other, the conjunction of "*B* in *A*" and there will remain: "*A* square 'plus' *B* in *A*" is equal to "*G* square 'plus' *D* plane."[36]

10. And if it happens that all the magnitudes have as a factor a certain rung, and therefore that the homogeneous element determined by the over-all measure does not immediately appear, there shall be a hypobibasm.[37]

Hypobibasm is the like lowering of the power and of the lower ladder magnitudes, the order of the ladder being observed, until the homogeneous element determined by the lower rung coincides with the over-all homogeneity according to which the magnitudes that remain are equated, by which operation the equation is not changed. But that must now be demonstrated.

The operation of hypobibasm differs from parabolism only in this, that in the case of hypobibasm each side of the equation is divided by an unknown quantity, but in the case of parabolism each side is divided by a known quantity, as is clear from the examples presented by the author.

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as a "length," and *e* as a "solid."

[35] "Antithesis" means the transposition of terms from one side of the equation to the other, with accompanying change of sign. Thus we might have the equation: "*A* cube + *A* square in *G* – *A* square in *H* – *B* plane in *C* + *F* plane in *K* is equal to *D* solid." By antithesis we could infer the equation: "*A* cube + *A* square in *G* – *A* square in *H* is equal to *B* plane in *C* – *F* plane in *K* + *D* solid."

In modern notation, from $x^3 + ax^2 - bx^2 - cd + ef = g$ we get by antithesis $x^3 + ax^2 - bx^2 = cd - ef + g$. We understand *c* and *e* as "planes" and *g* as a "solid."

[36] In modern notation, we are given that $x^2 - d = y^2 - bx$. We want to show that $x^2 + bx = y^2 + d$. We add to both sides $d + bx$ and get $x^2 - d + d + bx = y^2 - bx + d + bx$, or $x^2 + bx = y^2 + d$. Here *d*, of course, is understood as "*d* plane."

[37] "Hypobibasm" means dividing both sides of the equation by the unknown. It comes from the verb 'υποβιβαζω [hupobibazō], "to lower." Division "lowers" a magnitude from a higher rung to a lower rung.

Proposition II
An equation is not changed by hypobibasm

Let it be given that "*A* cube 'plus' *B* in *A* square" is equal to "*Z* plane in *A*."

I say that, by hypobibasm, "*A* square 'plus' *B* in *A*" is equal to "*Z* plane."

For that means to have divided all the "solids" by a common divisor, by which it is certain that the equation is not changed.[38]

11. And if it happens that the higher rung to which the unknown magnitude ascends does not subsist by itself but is multiplied by some given magnitude, parabolism[39] may be effected.

There is parabolism whenever the homogeneous magnitudes of which an equation is composed are divided by a given magnitude which in the equation appears as multiplied by the higher rung of the unknown magnitude, so that that rung assumes the name of the power, and in that power the final equation remains. But this must now be demonstrated.

Proposition III
An equation is not changed by parabolism

Let it be given that "*B* in *A* square 'plus' *D* plane in *A*" is equal to "*Z* solid."

I say that by parabolism "*A* square 'plus' $\frac{D\ plane}{B}$" is equal to "$\frac{Z\ solid}{B}$".

For that means to have divided all the "solids" by the common divisor *B*, by which it is certain that the equation is not changed.[40]

12. And then the equation shall be thought to be expressed clearly and shall be called "well ordered" (*ordinata*): it must be capable of being referred to a proportion, the following condition (*cautio*) especially being satisfied: the product of the extremes must correspond to the power together with the conjoined homogeneous elements; the product of the means must correspond to the homogeneous element under the given measure.[41]

13. Whence also an ordered proportion may be defined as a series of three or four magnitudes, so expressed in terms either simple or conjoined that all are given except that which is being sought, or else the power and the lower ladder magnitudes.[42]

[38] In modern notation: If $x^3 + bx^2 = cx$, then $x^2 + bx = c$. Here *c* is thought of as a "plane" so that *cx* is a solid; then it may be said that all the "solids" are divided by the common divisor *x*.

[39] "Parabolism" means dividing both sides of the equation by a known quantity. It comes from the verb παραβάλλω [paraballō] meaning "to apply," i.e., to divide, as when an area of *a* units is applied to a length of *b* units, the breadth of the figure will give the quotient *a/b*.

[40] In modern notation: If $bx^2 + cx = d$, then $x^2 + cx/b = d/b$, where *c* is thought of as a "plane" and *d* as a "solid."

[41] Thus, if we have the equation "*A* square + *B* in *A* is equal to *C* in *D* + *C* in *F*," then it follows that *A* is to *C* as "*D* + *F*" is to "*A* + *B*."

[42] That is, an ordered proportion would be one of the type given in the preceding note or one of the type $x^2 + ax : b :: c : d + x$, which yields the equation $x^3 + dx^2 + adx + ax^2 = bc$. This law and the preceding indicate that an ordered

14. Finally, when the equation has been thus ordered, or the proportion thus ordered, let it be considered that zetetics has performed its function.[43]

Diophantus in those books which concern arithmetic employed zetetics most subtly of all. But he presented it as if established by means of numbers and not also by species (which, nevertheless, he used), in order that his subtlety and skill might be the more admired; inasmuch as those things that seem more subtle and more hidden to him who uses the reckoning by numbers (*logistice numerosa*) are quite common and immediately obvious to him who uses the reckoning by species (*logistice speciosa*).[44]

Chapter VI
Concerning the investigation of theorems by means of the poristic art

When the zetesis has been completed, the analyst turns from hypothesis to thesis, and presents theorems of his own finding, theorems that obey the regulations of the art and are subject to the laws 'κατά παντὸς, καθ' αυτὸ, καθόλου πρωτον,'[45] which theorems, although they have from

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proportion is one that yields an equation in one unknown.

[43] The reader will recall the statement in Chapter I that it is the function of the Zetetic art to find the equation or proportion. In Viète's view, this is the most important part of the whole analytical art.

[44] As has been said in the preface, it was the opinion of quite a few people in Viète's day and later that the ancient mathematicians Apollonius, Pappus, and Diophantus actually knew algebra and used it to find the theorems they proved, but that they concealed it for reasons of vanity. In Rule IV of the *Rules for the Direction of the Mind* Descartes says:

> Indeed I seem to recognize certain traces of this true Mathematics in Pappus and Diophantus who, though not belonging to the earliest age, yet lived many centuries before our own times. But my opinion is that these writers then with a sort of low cunning, deplorable indeed, suppressed this knowledge. Possibly they acted just as many inventors are known to have done in the case of their discoveries, i.e., they feared that their method being so easy and simple would become cheapened on being divulged, and they preferred to exhibit in its place certain barren truths, deductively demonstrated with show enough of ingenuity, as the results of their art in order to win from us our admiration for these achievements, rather than disclose to us that method itself which would have wholly annulled the admiration accorded. Finally, there have been certain men of talent who in the present age have tried to revive this same art. For it seems to be precisely that science known by the barbarous name Algebra, if only we could extricate it from that vast array of numbers and inexplicable figures by which it is overwhelmed, so that it might display the clearness and simplicity which we imagine ought to exist in a genuine Mathematics.

Viète himself had written in the introduction to the *Ad Problema Adiani Romani Responsum*, "*Neque vero placet* barbarum idioma, *id est algebricum*." The Reverend John Wallis, in his preface to *A Treatise of Algebra*, says, "That it was in use of old among the Grecians we need not doubt; but studiously concealed by them as a great secret."

[45] Aristotle in the *Posterior Analytics* (I, 4, 73a21) says that we have demonstrative knowledge only when in the premises of the demonstration each predicate is "True of every instance of its subject" (κατά παντὸς [kata pantos]), is predicated "essentially" (καθ' αυτὸ [kath auto]) of its subject and is "commensurately universal" (καθόλου πρωτον [katholou prōton]) with its subject. Thus, the Euclidean definition of a circle—"A circle is a plane figure contained by one line such that all straight line falling upon it from one point among those lying within the figure are equal to one another"—fulfills all these conditions. For (1) there is no circle for which the predicate is not true of the subject; (2) the predicate belongs to the subject essentially, i.e., in virtue of what the subject is; and

the zetesis their demonstration and firmness, are subjected to the law of synthesis, which is considered a more logical way of demonstrating; and whenever there is occasion,[46] they are proved through it, [yet] by the great miracle of the art of finding. And for this reason, the steps of the analysis are retraced, which retracing is itself also analytical; and yet not in virtue of the reckoning by species (*logistice speciosa*), which has already performed its assigned duty. But if something unfamiliar has been hit upon and is proposed for proof, or if something has been presented by chance the truth of which must be weighed and investigated, then the way of poristic has first to be tried, from which it is easy to return to the synthesis; examples of this have been offered by Theon in the *Elements*, by Apollonius of Perga in the *Conics* and also by Archimedes himself in various books.[47]

Chapter VII
Concerning the function of the rhetic art

When the equation of the magnitude which is being sought has been set in order, the rhetic or exegetic (ρητικη η εξηγητικη) art, which is to be considered as the remaining part of the analytical art and as one which pertains principally to the application (*ordinationem*) of the art (since the two others are concerned more with general patterns (*exemplorum*) than with precepts, as one must by right concede to the logicians), performs its function both in regard to numbers if the problem concerns a magnitude that is to be expressed by number, and in regard to lengths, surfaces, and solids if it is necessary to show the magnitude itself. And, in the latter case, the analyst appears as a geometer by actually carrying out the work in imitation of the like analytical solution; in the former case, he appears as a logistician by resolving whatever powers have been presented numerically, whether simple powers or conjoined. Whether it be in arithmetic or geometry, he produces some specimens of his own [analytic] art according to the conditions of the equation that has been found or of the proportion that has been derived in an orderly way from it.

In fact, not every geometrical solution is a neat one, for particular problems have their

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(3) the predicate is commensurately universal with the subject as it is not, for example, with "figure" (for it is of less universality than "figure") or with "circle whose radius is less than ten inches" (for it is of greater universality than such a circle).

These rules, as applied to the propositions of "poristic" (for example, "*A* cube is equal to *B* plane in *C*") would seem to mean: (1) that the predicate must be "true of every instance" to which the subject is understood to refer, (2) that it must be predicated "essentially" of the subject, which would be the same as the law of homogeneity, and (3) that the predicate must be completely convertible with the subject as is the case when the predicate is "commensurately universal" with the subject. Regarding the last rule, we may remark that if one is to form the "synthesis" from the "analysis" by reversing or retracing the steps, each statement must be completely convertible.

[46] Alternatively, "Whenever the work is finished" (*si quando opus est*).

[47] It is difficult to see from this brief chapter in what way the poristic art differs from "synthesis," although in Chapter I, it is said to be part of the art of "analysis," which is distinguished from "synthesis." If we consider Propositions 44 et seq. of Bk. II of Apollonius' *Conics*, we find the method of analysis applied to the construction of diameters and tangents to conic sections. Each proposition has three parts: (1) the analysis proper: (2) a hypothetical statement which retraces the steps of the analysis and shows in a general way how the construction is to be carried out and (3) the actual construction, which Apollonius calls the "synthesis." It may be that Viète is thinking of "poristic" as having to do with the second of these. The examples that he gives in the *Supplementum Geometriae*, Propositions VI and VII, would seem to support this.

own elegances. But that solution is preferred to others which does not derive (*arguit*) the synthetic operation (*compositionem operis*) from the equation, but derives the equation from the synthesis, while the synthesis proves itself. Thus the skillful geometer, though a learned analyst, conceals this fact and presents and explicates his problem as a synthetic one, as if thinking merely about the demonstration that is to be accomplished; then, by way of helping the logisticians, he constructs (*concipit*) and demonstrates a theorem having to do with a proportion or an equation perceived in that synthetic problem.[48]

Chapter VIII
The symbolism (*notatio*) in equations and the epilogue to the art

1. In analysis the name equation is understood simply as referring to an equality properly set in order by means of the zetesis.

2. And so, an equation is the coupling (*comparatio*) of an unknown (*incertae*) magnitude with a known (*certa*).

3. The unknown magnitude is a root or power (*radix vel potestas*).

4. Again, a power is either simple or conjoined.

5. Conjunction exists either through subtraction or addition.

6. When an element homogeneous in conjunction is subtracted from a power, the subtraction is direct.[49]

7. When, on the contrary, the power is subtracted from the element homogeneous in conjunction, the subtraction is inverse.[50]

8. The measuring subrung is the measure itself of the rung of the element homogeneous in conjunction.[51]

9. But it is necessary to designate the rank of the power, the rank of the lower rungs, and also the quality or sign of the conjunction. Also the coefficient subrung magnitudes must be given.

10. The first lower ladder magnitude is the root which is being sought. The last is that which is lower than the power by one rung of the ladder. This is customarily understood by the name "epanaphora."[52]

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[48] The third part of the analytical art is perhaps not so much a part of the analytical art as it is an application of that art to the solution of particular arithmetical or geometrical problems. It is called rhetic (from ʻρεω [rheo] = "to flow forth, fall, say": cf. ρητος [rhetos] = "said, settled") when the application is arithmetical, i.e., when the solution is a number for which there is a word that can be said, as "three" or "sixty-one" or "nine-elevenths." It is called exegetic (from εξηγεομαι [exēgeomai], "to lead out, guide, interpret") when the application is geometrical, i.e., when the solution is a geometrical magnitude which can be shown to our imagination. Viète, however, uses the terms interchangeably of the arithmetical and geometrical applications. (Cf. Chapter VIII, 23 and 24.)

[49] For example, "*A* cube – *A* square in *G* – *A* square in *H*," or in modern notation, $x^3 - (gx^2 + hx^2)$.

[50] For example, "*A* square in *G* + *A* square in *H* – *A* cube," or in modern notation $(gx^2 + hx^2) - x^3$.

[51] This would seem to mean that in the case of "*A* cube – *A* square in *G* + *A* square in *H*," where "*A* square in *G* + *A* square in *H*" is the element homogeneous in conjunction, it is "*A* square'" which, while being the subrung of "*A* cube," measures the whole element "*A* square in *G* + *A* square in *H*."

[52] In modern notation, x^2 is the "epanaphora" of x^3, x^3 of x^4, x^4 of x^5, etc. The word "epanaphora" is from

Thus "square" is the epanaphora of "cube," "cube" of "squared-square," "squared-square" of "squared-cube," and so on in the same series in infinitum.

11. A lower ladder magnitude is the reciprocal of a lower ladder magnitude when a power is produced through the multiplication of one by the other. Thus, the coefficient magnitude is the reciprocal of that rung which it sustains.

As, for example, if there should be a "side" which is a lower ladder magnitude in relation to the "cube," the reciprocal rung will be the "square." But a "plane" multiplied by a "side" will be a reciprocal magnitude in relation to the "side," since the "solid" is produced from the "side" multiplied by the "plane," the "solid" being itself a magnitude of the same rung as the "cube."

12. After the root the lower ladder magnitudes progressing by "length" are the same ones that are designated on the ladder.

13. After the root the lower ladder magnitudes progressing by "plane" are:

"Square"	"*Plane*."
"Squared-square"	"Square of the *plane*."
"Cubed-cube"	"Cube of the *plane*."

And so on successively in that order.

14. After the root the lower ladder magnitudes progressing by "solid" are:

"Cube"	"*Solid*."
"Cubed-cube"	"Square of the *solid*."
"Cubed-cubed-cube"	"Cube of the *solid*."

15. "Square," "Squared-square," "Squared-cubed-cube," and those magnitudes which are produced from these continuously in this order are simple middle powers; the rest are manifold.

Thus, the simple middle powers can also be defined in such a way that they will be those the exponents of which progress in the geometrical subduplicate ratio. So powers of the second degree, of the fourth, of the eighth, of the sixteenth, will be simple middle powers. The remaining powers, standing in the intermediate degrees, are manifold.[53]

16. A known magnitude with which the others are equated is the homogeneous element of the equation.

As, for example, if "A cube + A in B square is equal to B in Z plane," "B in Z plane" will be the homogeneous element of the equation.

"A cube" will be the power (potestas) *to which the unknown magnitude which is being sought ascends by its own nature* (vi sua).

"A in B square" will be the element homogeneous in conjunction.

"A" is the lower ladder magnitude.

"B square" is a subrung magnitude or "parabola."[54]

επαναφέρω [epanapherō], "to earn up to, refer to, report to."

[53] If of the series of ladder magnitudes we consider x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}, x^{12}, x^{13}, x^{14}, x^{15}, x^{16}, ... and then the exponents of the powers x^2, x^4, x^8, x^{16}, ..., we see that the exponents are in geometrical progression: 2:4 :: 4:8 :: 8:16. They may be said to progress in the subduplicate ratio in that 2:4 is the subduplicate of 2:8, 4:8 is the subduplicate of 4:16, etc.

[54] "Parabola" here means the result of application or the quotient resulting from division by the unknown.

17. In the case of numbers, the homogeneous elements of equations are units.[55]

18. When a "root" that is being sought is, while remaining on its own base, equated to a given homogeneous magnitude, the equation is simple absolutely.[56]

19. When the power of a "root" that is being sought, being free from all conjunction, is equated to a given homogeneous magnitude, the equation is simple ladder-wise.[57]

20. If the power of a "root" that is being sought is joined with magnitudes at the designated rung accompanied by their given coefficients and if it is equated to a given magnitude, the equation is polynomial in proportion to the multitude and variety of the conjunction.[58]

21. A power can be involved in as many conjunctions as there are ladder magnitudes lower in relation to that power.

Thus, a "square" can be conjoined with a magnitude at the rung of the "side"; a "cube" with magnitudes at the rungs of the "side" and the "square"; a "squared-square" with magnitudes at the rungs of the "side," the "square," and the "cube"; a "squared-cube" with magnitudes at the rungs of the "side," the "square," the "cube," and the "squared-square"; and so on in that series *in infinitum.*

22. Proportions are distinguished from one another and receive their nomenclature from the kinds of equations into which they are resolved.

23. With a view to exegetic in arithmetic the trained analyst is taught:

To add a number to a number.

To subtract a number from a number.

To multiply a number by a number.

To divide a number by a number.

The art, furthermore, yields the resolution of all possible powers whether they be pure or (a thing of which both ancients and moderns have been ignorant) conjoined [with other magnitudes].[59]

24. With a view to exegetic in geometry the trained analyst selects and enumerates more regular procedures (*recenset effectiones magis canonicas*) by which equations of "sides" and

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[55] In the equation "*A* cube is equal to *B* plane in *C*," the homogeneous element of the equation is "*B* plane in *C*" and its unit is a "solid" unit. We could not have "*A* cube is equal to *B* plane," for the unit of "*B* plane" is a "plane" unit. It is different with equations involving numbers rather than species. In the equation "*A* is equal to 9," "*A*" becomes a number, and the units of all numbers are the same in kind as long as the numbers are pure numbers and not numbers of apples or elves or what not.

[56] "*A* is equal to *B*," or $x = a$ in modern notation.

[57] "*A* cube is equal to *B* solid," or $x^3 = a$ in modern notation, a being understood as a "solid."

[58] "*A* cube + *B* in *A* square – *C* plane in *A* is equal to *D* solid," or, in modern notation, $x^3 + ax^2 - bx = c$, where b is a "plane" and c is a "solid."

[59] This is the program of Viète's work *De Numerosa Potestatum Purarum, atque Adfectarum ad Exegesin Resolutione* (*Opera Mathematica*, p. 163, cf. Note 210). The "numerical resolution of powers" means the solution of equations that have numerical solutions, such as the equation $x^2 = 2916$ (Problem 1, p. 165, of the first section of the *De Numerosa*, which section has to do with pure powers), or $x^2 + 7x = 60{,}750$ (Problem 1, p. 174, of the second section, which has to do with conjoined powers).

"squares" may be completely interpreted.[60]

25. With a view to "cube" and "squared-square," in order that the deficiency of geometry may be supplied as if (*quasi*) by geometry, the analytical art postulates that

A straight line can be drawn [ducere] *from any point across any two lines in such a way that the intercept between these two lines will be equal to a given distance, any possible intercept having been predefined.*

This being granted (it is, indeed, a postulate not difficult to fulfill), analysis skillfully solves the more famous problems which have hitherto been called irrational: the mesographicum, the problem of the division of an angle into three equal parts, the finding of the side of the heptagon, and as many others as fall into the formulas of equations in which "cubes" are equated to "solids," "squared-squares" to "plane-planes," whether simply or with some conjunction.[61]

26. Since all magnitudes are lines, surfaces, or solids, what great use could be made in human affairs of proportions involving triplicate or even quadruplicate ratios, if not perhaps in divisions of angles, so that we might obtain the angles from the sides of the figures or the sides from the angles?

27. Therefore, analysis, whether with a view to arithmetic or to geometry, discloses the mystery, known hitherto by no one, of the division of angles, and it teaches how:

When the ratio of the angles is given, to find the ratio of the sides.
To make an angle to be in the same ratio to an angle that a number is to a number.

28. It does not equate (*comparat*) a straight line to a curve, because an angle is something in between a straight line and a plane figure. Thus, the law of homogeneity seems to oppose it.

29. Finally, the analytical art, having at last been put into the threefold form of zetetic, poristic, and exegetic, appropriates to itself by right (*iure*) the proud problem of problems, which is:

TO LEAVE NO PROBLEM UNSOLVED.[62]

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[60] This is the program of Viète's *Effectionum Geometricarum Canonica Recensio* (*Opera Mathematica*, pp. 229 ff), at the beginning of which he says, "The geometrical procedure by which all equations which do not exceed the measure of squares may be rightly interpreted I enumerate as follows ...."

[61] This is the program of Viète's *Supplementum Geometriae* (*Opera Mathematica*, pp. 240 ff.), which begins with a restatement of the postulate about the intercept and which contains Viète's solutions of the three problems here mentioned. Propositions V-VII of the *Supplementum Geometriae* contain the solution of the problem of the mesographicum; this was the problem of finding two mean proportionals to two given straight lines, and its solution immediately yields the solution of the problem of doubling the cube. Proposition IX contains the solution of the problem of the trisection of an angle. Proposition XXIV contains the solution of the problem of finding the side of the regular heptagon which is to be inscribed in a given circle.

[62] This proud claim for the new Mathematics, of which Viète more than anyone else was the founder, is characteristic not only of Viète, but of Descartes and Newton and, one may say, of the whole modem age. The capital letters are Viète's.

FRANCISCI VIETAE

EFFECTIONUM GEOMETRICARUM

Canonica recensio

FRANÇOIS VIÈTE

Standard Enumeration of

GEOMETRICAL RESULTS

Translated by

Richard Ferrier

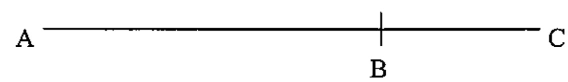
THE STANDARD ENUMERATION OF GEOMETRICAL RESULTS[1]

Those geometrical results by means of which all equations which do not exceed the bounds set by squares may be conveniently explicated, I list in standard order as follows:[2]

PROPOSITION I

To 'add' [*addere*] a given straight line to a given straight line.[3]

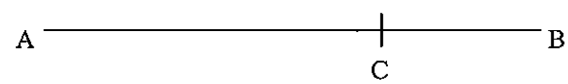
Operation of addition. Let the two given straight lines be AB, BC. It is required to 'add' the one to the other. Let AB be extended by length BC. I say that what was required has been done. For AC is composed of AB, BC.



PROPOSITION II

To 'take away' [*auferre*] a given straight line from a greater, given straight line.

Operation of subtraction [subductionis]. Let the two given unequal straight lines be AB, BC. It is required to 'take away' the lesser from AB, the greater. Let BC be cut off from AB. I say that what was required has been done. For AC is the difference between AB and BC.

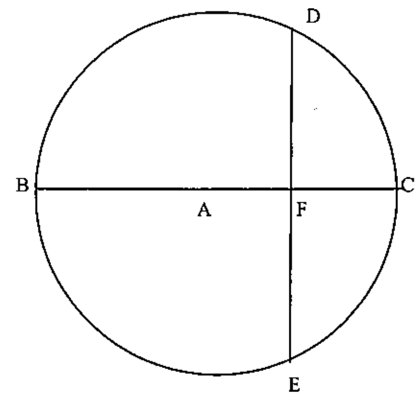


[1] The title might also be rendered, "a standard edition of geometrical constructions." It is with reference to this geometrical explication of certain equations and the subsequent determination of the unknown in each that this selection of constructions becomes "canonical" or standard.

[2] As the propositions XIV-XX show (and which are not included in this translation), this advertisement is misleading, since equations of square-squares (though not of cubes) can be interpreted by geometrical constructions. These equations are resolved, however, from proportions between squares, and in this sense they do not "exceed the bounds set by squares." Cf. *Isagoge,* ch. 8, no. 24.

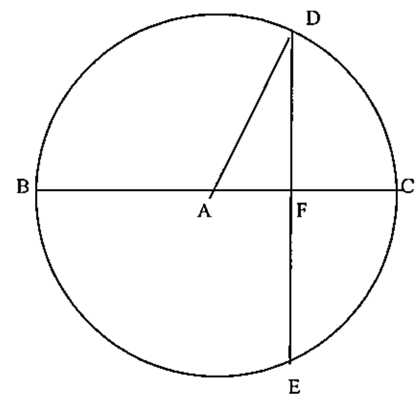
[3] The translator has placed the word 'add' in quotation marks since this proposition serves to define geometrically the operation of addition of species that are of the first "rung" or genus, i.e., sides, lengths, and widths. That is, just as the species are interpreted as given lines, so the sign '+' or the word 'plus' is interpreted by the construction indicated in the proposition (cf. *Isagoge*, ch. 3, nos. 2-4; and ch. 4, Precept I). The proof of the proposition consists in showing that the sense of 'to add' in specious logistic is analogous to the sense of 'to add' in geometry as here defined, in that the sum is composed of the parts in both. Note, however, that the lines are here given, whereas in specious logistic the magnitudes need not be known. Thus, 'A + B' (the sum of a known and an unknown) should not be interpreted immediately by this proposition (although Viète does use it for this purpose), whereas 'B + D' (the sum of two knowns) may be.

PROPOSITION III
To draw three proportional straight lines.



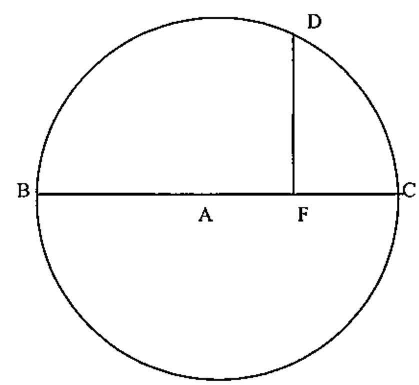
About center A, with any distance, let a circle be described, and the diameter BAC be drawn. Let equal [arcs] CD, CE be taken in opposite parts of the circumference, and let DE, when joined, cut BC in F. I say that what was required has been done. For BF, FD, FC are proportional.[4]

PROPOSITION IV
To draw a right triangle.



With the construction above repeated, let AD be joined. I say that what was required has been done. For AFD is a triangle, and it is right, since angle AFD is right, as is demonstrated in the *Elements*.[5]

PROPOSITION V
Given two straight lines, to find the mean proportional between them.



The operation of multiplication [multiplicationis]. *For multiplication is:* given sides, to find a plane, or to exhibit a square equal to that plane. The tradition holds,[6] moreover, that the plane made by the extremes is equal to the square on the mean.[7]

Let the two given straight lines be BF, FC. It is required to find a mean proportional between them. Let BF be extended by the length FC, and let BC be bisected at A. Let a circle be drawn with center A and radius AB or AC, and from point F, let a perpendicular be erected cutting the circumference at D. I say that what was required has been done. For DF is the mean which was sought, as is obvious from the standard [*canonica*] drawing of the three proportionals.

Thus the square equal to a given plane is itself given.[8]

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[4] That is, BF : FD :: FD : FC. Cf. Euclid, *Elements* VI, 13.

[5] This is nowhere enunciated in the *Elements*, though it is an easy inference from III, 26. Proclus calls problems like this and the preceding "deficient" (that is, insufficiently determinate, since there exist indefinitely many figures which solve the problem), and praises Euclid for avoiding such a form in the first problem (I, 1) in the *Elements*. See Proclus, *Commentary*, pp. 221-222.

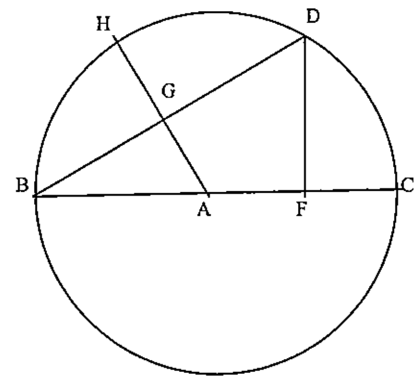
[6] I.e., the geometrical tradition. Literally, "It has been handed down, moreover. . ." (Lat., *Traditum est autem* . . .)

[7] Euclid, *Elements* VI, 17.

PROPOSITION VI

Given two straight lines, to find a third proportional.

The operation of division [adplicationis]. *For that is:* to apply [*adplicare*] a given plane, or a square equal to a plane, to a straight line, and to show the width[9] which arises. *To wit, the square of the mean is applied to the first, and the third arises.*[10]



Let the two given straight lines be BF, FD. It is required to find the third proportional. Let BF, FD lie at right angles, and let BD be joined. Let BD be bisected at right angles by the straight line AH, cutting BD at G and BF at A. And let a circle be drawn, with center A and radius AB or AD, to the circumference of which let BF be produced, meeting it at C. I say that what was required has been done. For FC is the third proportional, which was sought, to the given lines BF, FD, as is obvious from the standard drawing of three proportionals.

The following constructions [effectiones], *however, are less standard.*

1. Given three straight lines, to find the fourth proportional.
2. To make a line to a line as a number to a number, with one line being sought, while all the rest are given.
3. To make a line to a line as a square to a square, with one line being sought, while all the rest are given.
4. To make a square to a square as a line to a line, with the side of one square unknown, while all the rest are given.

Even so, if these results should ever be needed, they may be had from the *Elements*.[11]

The following constructions, on the other hand, are not only altogether regular, they are also to be recommended for their frequent use and employment.

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[8] This proposition is included for two reasons: 1) It defines "to multiply" (in the first genus) for geometry as "to make a rectangle," and 2) it enables the rectangles to be compared and ordered by finding equivalent squares. Viète seems to argue that since a side determines its square, and a square can be found equal to the rectangle, to find the side is to find the rectangle. Cf. Euclid, *Elements* II, 14.

[9] The term "to apply" (παράβαλλειν [paraballein] in Euclid) means to find the width that, together with a given length, forms a rectangle equal to a given area. This given area is then said to have been "applied" to the given length, and the width is said to "arise" from the application. (See Heath, *Commentary on the Elements*, 1: 343-345 and 374.) In the *Canonica Recensio* Viète consistently uses the term *latitudo* for the line arise from the application; it is rendered as "width" throughout this translation.

[10] Cf. Euclid, *Elements* I, 44, and also Proclus, *Commentary*, 419.15-420.23. In the terms of specious logistic, suppose B plane to be divided by D. There is no difficulty in finding F square equal to B plane. Set F square / D = A. Then D : F :: F : A. Thus, in specious logistic to find A, the third proportional, is equivalent to division.

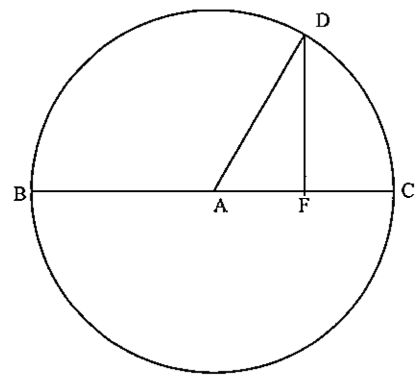
[11] Euclid, *Elements* VI, 9; VI, 12; and VI, 19, porism.

PROPOSITION VII

Given the two sides about the right angle of a right triangle, to find the third side.

The operation of addition of planes. For, in truth, Pythagoras has taught that, *the squares on the sides about the right angle are equal to the square on the remaining side.*

And this same thing the analytical principles, too, argue from the same drawing of the triangle. For it has been handed down from analytics that the sum of two sides, when multiplied into[12] the difference of the sides, makes the difference of the squares. But the sum of AD, or BA, and AF is BF, and the difference between AD, or AC, and AF is FC. Moreover, BF multiplied into FC makes the square on DF. Thus the square on DF is the difference between the square on AD and that on AF. And by the change of sides of the analytic art, which is called "antithesis,"[13] the square on AD is the sum of the squares on AF and DF.

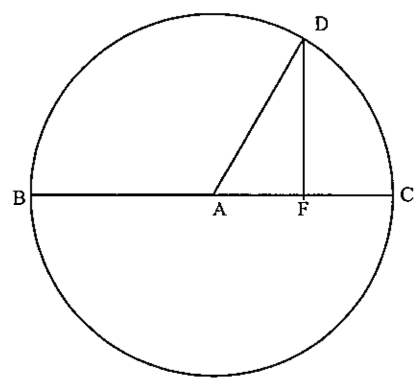


Let the two given sides about the right angle of the right triangle be AF, FD. It is required to find the third side, the one which subtends the right angle. Let AF, FD lie at right angles, therefore, and let AD be joined.

I say that what was required has been done. For AD is the side which was sought, subtending DFA the right angle of the triangle, which angle was made by AF, FD.[14]

PROPOSITION VIII

Given the side subtending the right angle of a triangle and one of the remaining sides, to find the third side.



The operation of subtraction of planes. Let there be given two sides of a right triangle: one, AC, subtending the right angle, the other, AF, situated about the right angle. It is required to find the remaining side. With center A and radius AC let a circle be drawn, let AF be cut off from AC, and let a perpendicular be drawn to AC at point F, let this cut the circumference at D, and let AD be joined.

I say that what was required has been done. For DF is the side which was sought, bounding the right angle in triangle AFD, whose remaining sides AF and AD, that is, AC, were given.

[12] Lat.: *ducitur in, ducta in.* Here translated as "multiply into," but when "in" appears alone, translated as "times" (see note 15).

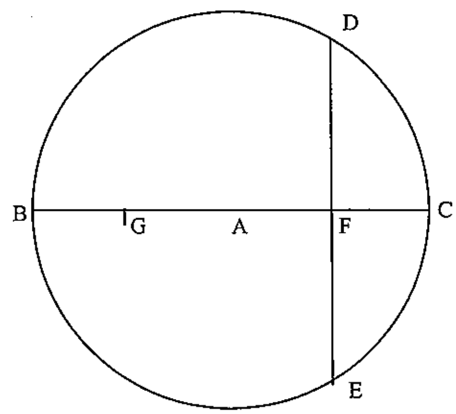
[13] Lat.: *per artis translationem, quae dicitur antithesis.* See *Isagoge*, 5, no. 1. "Antithesis" is the transposition of a quantity from one side of an equation to the other.

[14] The chief aim of this proposition is not, of course, the painfully obvious construction, but the interpretation of "to add" (in the second genus) geometrically (viz., the addition of planes). Beyond this, Viète demonstrates the Pythagorean Theorem as an easy deduction from specious logistic, perhaps to indicate the superiority of the analytic art to ordinary geometry. It is for this purpose alone that the full diagram is needed.

PROPOSITION IX

Whenever there are three proportional straight lines, the square on the smaller extreme together with the rectangle contained by the difference of the extremes and this same lesser extreme is equal to the square on the mean.

Let the standard diagram of the three proportional straight lines be set out, and let FC be understood to be the lesser extreme, and let BG be laid out equal to it. Hence the difference between BF, the greater extreme, and BG, that is FG, the lesser extreme, is FG.



I say that the square on CF together with the rectangle contained by CF, FG is equal to the square on DF. For the square on CF is alternatively made from CF times GB.[15] And thus the two products CF times BG and CF times FG are equivalent to the product CF times FB. But since this product is the rectangle contained by the extremes, the square on the mean between the extremes, DF, is equal to it.

COROLLARY FOR THE MECHANICS[16]

of a square affected by joining a sublateral plane

Therefore, whenever A square plus B in A is proposed equal to D square: D will be understood as the mean between extremes, and B their difference. And from the mean and the difference of the extremes, the extremes will be sought, the lesser of which will be A, the unknown.

As here, from given lines GF, FD, the proportionals BF, FD, FC will be constructed. And FC will be the lesser, which was sought. As could have been argued from Zetetics,[17] and here the

[15] Lat., *aliter est factum ex CF in GB*. I render this and similar phrases "CF times GB," though "into," "on," or "over" might convey the sense more exactly. See note 12.

[16] Lat. *Consectarium ad*, which could also be translated as "consequent to." *Mechanice* is the Latinized form of μηχανική (the feminine indicates that τέχνη [technē], "art," is understood), meaning "the methodical or artful procurement or invention of something." In the *Canonica recensio*, it means "the standard way to do" a certain algebraic problem geometrically. In the present case, "the Mechanics of a square affected by joining a sub-lateral plane," would mean, "the technical procedure for solving geometrically an equation in which the 'homogeneous element of the comparison' (*Isagoge* 5, no. 6) is equated with a square plus a sub-lateral plane (D square = A square + B in A plane)." It should be noted that finding a "mechanice" for any equation of the second degree (and some of the fourth) is the purpose of this treatise.

[17] The zetetic argument would run as follows:

$A^2 + AB = D^2$.

$A(A + B) = D^2$.

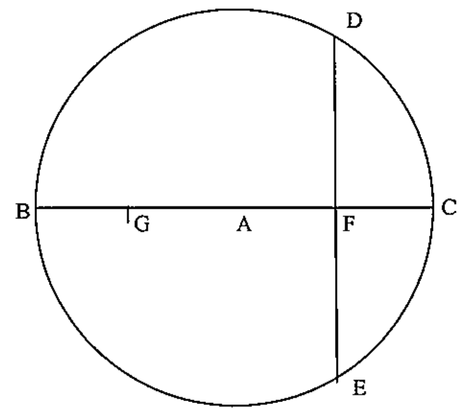
$A : D :: D : A + B$.

Thus A is the lesser extreme, D the mean, B the difference between the extremes. Cf. *Zetetica* 3, Prop. I.

Geometrical figure demonstrates through synthesis.[18]

PROPOSITION X

Whenever there are three proportional straight lines, the square of the greater extreme minus the rectangle contained by the difference of the extremes and the greater extreme is equal to the square of the mean.



Let the immediately preceding construction be repeated.

I say that the square on BF, minus the rectangle contained by BF, GF is equal to the square on DF. For the square on BF is equivalent to the product BF times GF, and, in addition, BF times BG. From the square on BF, therefore, let the product BF times FG be taken away, and the product BF times BG will remain, that is, by construction, BF times FC. But, since this product is the rectangle contained by the extremes, the square on the mean between the extremes, DF, is equal to it.

COROLLARY FOR THE MECHANICS OF A SQUARE
affected by removing a sublateral plane

And so, whenever A square less B in A is proposed equal to D square, D will be understood as the mean between extremes, B as their difference. And from the mean and the difference of the extremes, the extremes will be sought, the greater of which will be A, the unknown.

As here from given lines GF, FD the proportionals BF, FD, FC will be constructed. And BF will be the greater extreme, which was sought. As could have been argued from Zetetics,[19] and here the Geometrical figure demonstrates through synthesis.

PROPOSITION XI

Whenever there are three proportional straight lines, the rectangle contained by the suit, of the extremes and either one of them, the greater or the lesser, minus the square of this extreme is equal to the square of the mean.

Let the standard diagram of the three proportionals be set out. I say that the rectangle contained by BG, FC less the square on FC is equal to the square on DF.

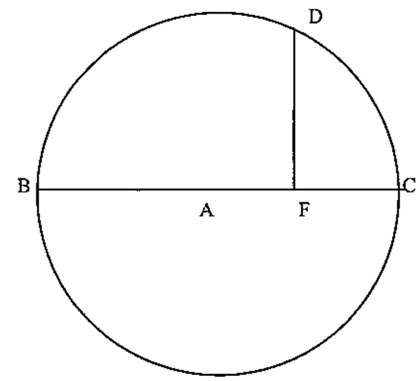
And again, the rectangle contained by BC, BF minus the square on BF is equal to the

[18] Note that the proposition does *not* teach how to find FC, given GF, FD. Rather, it demonstrates a geometrical theorem which explicates $A^2 + AB = D^2$. The geometrical analog of the algebraic solution will consist in actually deriving FC (or A, the unknown) in a construction which begins with GF (the difference between extremes, or B) and FD (the mean, or D) as given. This construction, which will be a problem, not a theorem, Viète gives in Prop. XII.

[19] Cf. *Zetetica* 3, Prop. I.

square on DF.

For, since BC is composed of BF, FC, therefore the product BC times FC is as much as the product BF times FC and FC times FC (that is, the square on FC). And thus when the square on FC is taken away from BC times FC, the product BF times FC will remain. But since this product is the rectangle contained by the extremes, the square on the mean between extremes, DF, is equal to it. And let this be the first case.



Likewise, since BC is composed of CF, FB, therefore the product BC times BF is as much as the product CF times BF and BF times BF (that is, the square on BF). And thus when the square on BF is taken away from the product BC times BF, the product CF times BF will remain. But since this product is the rectangle contained by the extremes, the square on the mean between the extremes, DF, is equal to it. As was to be shown in the second instance.

COROLLARY FOR THE MECHANICS OF A SUBLATERAL PLANE
diminished by a square

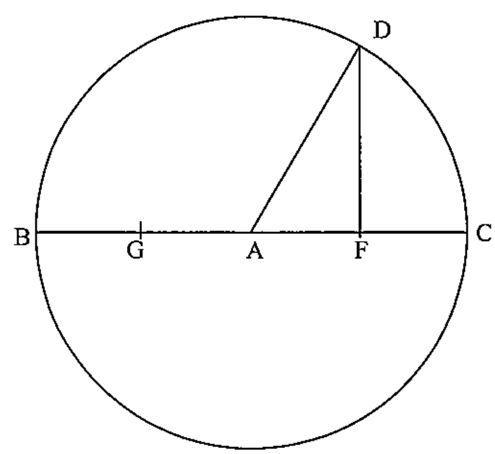
And so, whenever B in A less A square is proposed equal to D square, D will be understood as the mean between extremes, B as their sum. And from the mean and the sum of the extremes the extremes will be sought, either one of which will be A, the unknown.

As could have been argued from Zetetics,[20] and here the Geometrical figure demonstrates, through synthesis.

PROPOSITION XII
Given the mean of three proportionals and the difference of the extremes, to find the extremes.

The mechanics of a square affected by a sublateral [plane].

Let FD, the mean of the three proportionals, and also GF, the difference of the extremes, be given. It is required to find the extremes. Let GF, FD lie at right angles, and let GF be bisected at A. Then with center A and radius AD, let a circle be drawn, and let AG, AF be extended to its circumference at points B, C.



I say that what was required has been done. For BF, FC have been found, the extremes whose mean proportional is FD. And these same lines BF, FC differ by FG, since AF and AG were constructed equal, AC and AB were also constructed equal, and subtracting equals AG, AF from equals AB, AC, the remainders BG, FC are also equals. GF, however, is the difference

[20] Cf. *Zetetica* 3, prop II.

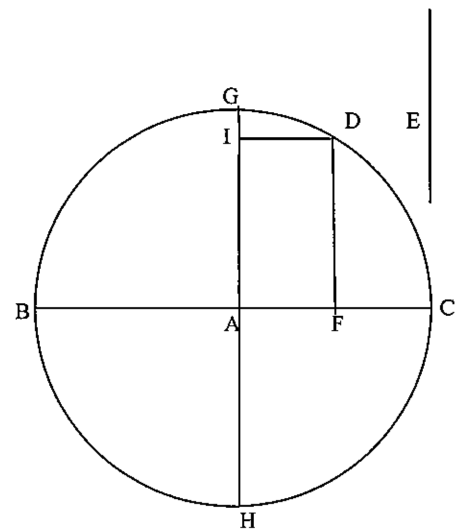
between BF and BG, or FC. Which was to be shown.[21]

PROPOSITION XIII

Given the mean of three proportionals and the sum of the extremes, to find the extremes.

The mechanics of a sublateral plane diminished by a square.

Let E, the mean of the three proportionals, and BC, the sum of the extremes, be given. It is required to find the extremes. Let BC be bisected at A, and with center A and radius AB or AC let a circle be drawn. And let another diameter GAH cut BAC at right angles, and let AI, equal to E, be cut off from AG. Next, let a straight line be produced through I, parallel to BC and intersecting the circumference in point D, from which point let DF fall perpendicular to BC, parallel to IA, and equal to it.



I say that what was required has been done. For the extremes which were sought are BF, FC, from which the given, BC, is composed. And DF, or IA, that is, E, becomes the mean between the proportionals.

. . . . .

[21] This problem solves for the unknown 'A' in the equations explicated by Props. IX and X, i.e., $A^2 + AB = D^2$ and $A^2 - AB = D^2$. 'A' is interpreted in Prop. IX as one extreme and in Prop. X as the other, while 'B' and 'D' receive the same interpretation in both propositions.

APPENDIX
COMPARISON OF THE METHODS OF DIOPHANTUS AND VIÈTE

Diophantus, *Arithmetic* - Book I, Problem I
To divide a given number into two number with a given difference

Let the given number be 100, and the difference be 40.

To find the numbers, let the lesser be taken as x.[22] Then the greater will be x + 40. Then both together become 2x + 40. But they have been given as 100. 100, then, are equal to 2x + 40. And taking like things from like: I take 40 from 100 and likewise 40 from 2x and 40. The 2x are left equal to 60. Then each x becomes 30. As to the actual numbers required: the lesser will be 30 and the greater 70, and the proof is clear.

Viète, *Zetetics* - Book I, Problem 1
Given the difference of two *sides* and their sum, to find the sides.

Let the difference, B, of two *sides* be given, and also their sum, D, be given. It is required to find the *sides*.

Let the lesser *side* be A; then the greater *side* will be A + B. Therefore, the sum of the *sides* will be 2A + B. But the sum is given as D. Therefore, 2A + B is equal to D. And, by antithesis, 2A will be equal to D – B, and if they are halved, A will be equal to ½ D – ½ B.

Or, let the greater *side* be E. Then the lesser will be E – B. Therefore, the sum of the *sides* will be 2E – B. But the same sum is given as D. Therefore, 2E – B will be equal to D, and by antithesis, 2E will be equal to D + B; and if they are all halved, E will be equal to ½ D + ½ B.

Therefore, with the difference of the two *sides* given and their sum, the sides are found.

For indeed, half the sum of the sides minus half their difference is equal to the lesser side, and half their sum plus half their difference is equal to the greater.

Which very thing the zetesis shows.

PROBLEM

Solve Diophantus's problem using Viète's method; i.e., let B = 40, D = 100, and solve for A and E.

[22] Actually, Diophantus uses some strange symbol for the unknown, and Greek letters for the knowns. The notation has been modernized for ease of understanding.

Diophantus, *Arithmetic* - Book I, Problem 4
To find two numbers in a given ratio so that their difference is also given.

Let the greater be 5 times the lesser, and let their difference be 20. Let the lesser be x; then the greater will be 5x. I require, moreover, that the 5x exceed x by 20; but let this excess be 4x. These are equal to 20. The lesser number will be 5, and the greater 25. For the lesser is 5 times the lesser, and the difference is 20.

Viète, *Zetetics* - Book I, Problem 2
Given the difference of two sides, and their ratio, to find the sides.

Let B be the given difference of the two *sides*, and R to S the given ratio of the lesser to the greater *side*. It is required to find the *sides*.

Let the lesser *side* be A. Therefore, the greater *side* will be A + B. Hence, A is to A + B as R is to S. Which proportion having been resolved, S times A will be equal to R times A + R times B. And by transposition under the opposite sign of conjunction, S times A – R times A will be equal to R times B, and, when all are divided by S – R, $\frac{R \times B}{S - R}$ will be equal to A. Wherefore, as S – R is to R so is B to A.

Or, let the greater *side* be E. Therefore, the lesser *side* will be E – B. Hence, E is to E – B as S is to R. Which proportion having been resolved, R times E will be equal to S times E – S times B. And by a suitable transposition, S times E – R times E will be equal to S times B. Wherefore, as S – R is to S so is B to E.

Therefore, given the difference of two *sides* and their ratio, the *sides* are found.

For indeed, as is the difference of the two sides, S and R to one of them, either the greater or the lesser, so is the given difference to either the greater or the lesser of the sides that are to be found.

PROBLEM
Solve Diophantus's problem using Viète's method.

René Descartes

The Geometry

Published in 1637

Translated by John Francis Nieto

With the method for finding the perpendicular translated by R. Glen Coughlin

and minor revisions by Christopher A. Decaen

The Geometry[1]

First Book

Problems which can be constructed without employing anything but circles and straight lines.

All the problems of geometry can be easily reduced to such terms that there is no need beyond that of knowing the length of certain straight lines to construct them.[2]

How the calculation of arithmetic relates to the operations of geometry.

And as all arithmetic is composed of only four or five operations, which are addition, subtraction, multiplication, division, and the extraction of roots, which can be taken for a species of division,[3] so there is nothing else to do in geometry concerning the lines that are sought, to prepare them to be known, than adding or subtracting others [to or from] them; or else having one, which I shall call the unit to relate so much more to numbers, and which can ordinarily be taken at one's discretion, then having yet two others, to find a fourth, which would be to one of the two as the other is to the unit, which is the same as multiplication; or else to find a fourth, which would be to one of these two, as the unit is to the other, which is the same as division; or finally to find one or two or more mean proportionals between the unit and some other line, which is the same as to draw out the square root or cube root, etc. And I do not fear to introduce these terms from arithmetic into geometry, to render myself more intelligible.

Multiplication.

Let, for example, AB be the unit and let it be necessary to multiply BD by

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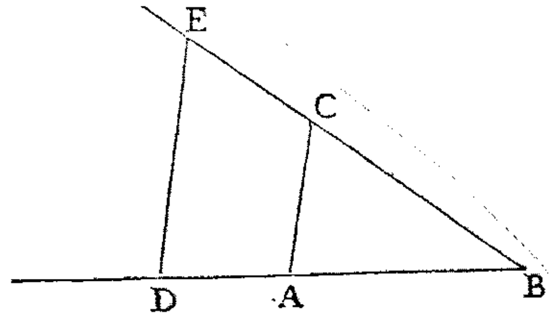
[1] *The Geometry* was originally published as one of three appendices to the *Discourse on Method*. The other two appendices (or "attempts with this method" [*essais de cette Méthode*], as Descartes described them; see also the top of p. 55 for this same expression) were the *Dioptrics* and the *Meteorology*.

[2] It may be noted at the start that *The Geometry* addresses itself to problems. The student may wish to consider what the difference between a problem and a theorem is. (Cf. Heath, *Euclid's Elements*, Vol. 1, pp. 124-129.) Descartes does not defend his claim in this treatise, and perhaps the reader will doubt whether it holds universally. For an example of a problem whose solution does not clearly require the knowledge of the length of some straight line, consider the inscription of a regular pentagon in a circle (*Elements* IV, 10 & 11).

[3] Is Descartes taking arithmetic to mean the totality of arithmetic problems here? What about arithmetic theorems? Or is the problem/theorem distinction not applicable to arithmetic? This last question suggests another, namely, How closely related are geometry and arithmetic? (Cf. *Rules for the Direction of the Mind*, Rule IV.)

BC, I have only to join the points A and C, then to draw DE parallel to CA, and BE is the product of this multiplication.[4]

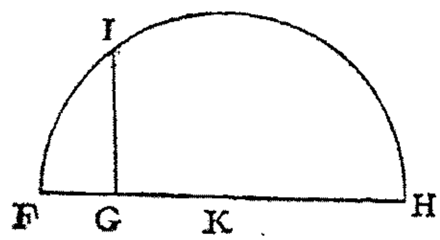


Division.

Or else if it is necessary to divide BE by BD, having joined the points E and D, I draw AC parallel to DE, and BC is the product of this division.

The extraction of the square root.

Or else if it is necessary to draw the square root of GH, I add FG, which is the unit, to it in a straight line, and dividing FH in two equal parts at the point K, from the center K, I draw the circle FIH, then setting up from the point G a straight line so far as I, at right angles to FH, GI is the root sought. I say nothing here of the cube root, nor of others, because I shall speak of them more conveniently later.



How one can use ciphers in geometry.[5]

But often one has no need to trace the lines thus on paper and it suffices to designate them by some letter, each by only one. So to add the line BD to GH, I name the one a and the other b, and I write $a + b$; and $a - b$, to subtract b from a; and ab, to multiply them, the one by the other; and a/b, to divide a by b; and aa, or a^2, to multiply a by itself; and a^3, to multiply it once more by a, and thus to infinity; and $\sqrt{a^2 + b^2}$, to draw the square root of $a^2 + b^2$; and $\sqrt[3]{a^3 - b^3 + abb}$, to draw the cube root of $a^3 - b^3 + abb$, and thus with others.

Here it is to be noted that by a^2 or b^3 or the like, I ordinarily conceive only lines altogether simple, although, to use the names employed in the algebra, I name them square or cubes, etc.

It is also to be noted that all the parts of one and the same line ought ordinarily to be expressed by as many dimensions, the one as the other, when the unit is not at all determined in the question, as here a^3 contains as many as abb or b^3 of which the line which I have named $\sqrt[3]{a^3 - b^3 + abb}$ is composed; but that it is not the same when the unit is determined, because it can be understood everywhere where there are too many or too few dimensions; so if

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[4] On Descartes's method of multiplication, see Exercises, p. 84.

[5] The reader may find it illuminating at this point to consult the *Oxford English Dictionary* for the meaning(s) of "cipher" (here translating "chiffre"), and also "symbol." Indeed, this and the previous section are greatly illuminated by *Rules for the Direction of the Mind*, rule 16.

it is necessary to draw the cube root from *aabb* - *b*, it is necessary to think that the quantity *aabb* is divided once by the unit and that the other quantity *b* is multiplied twice by the same.[6]

Moreover, in order not to fail to remember the names of these lines, a separate register must always be made to the extent that one posits them or changes them, writing, for example,

AB = 1, that is to say, AB equals 1.

GH = *a*

BD = *b*, etc.

How it is necessary to arrive at Equations which serve to resolve problems.

Thus wishing to resolve some problem, one must first consider it as already done and give names to all the lines which seem necessary to construct it, to those which are unknown as well as to the others. Then without considering any difference between the known lines and the unknown, we must go over the difficulty, according to the order which shows most naturally of all in what manner they depend mutually on each other, until we have found means of expressing one same quantity in two ways: what is named an equation, for the terms of one of these two ways are equal to that of the others. And we must find as many such equations as the lines we have supposed, which are unknown. Or else if so many are not found and yet we have omitted nothing of what is desired in the question, this testifies that it is not entirely determined. And then we can take known lines at our discretion for unknown lines to which no equation corresponds. After this if there remain still more,[7] we must use in order each of the equations which so remain, either by considering each alone or by comparing it with the others, to explain each of the unknown lines, and so disentangle them that there remains but one, equal to another, which is known, or else whose square or the cube or the square of the

[6] Descartes here uses the term "dimension" not to distinguish the line from the plane, but to distinguish a line from a line, depending on the generation of those lines. Thus, if *a* and *b* are lines, *ab* is a line generated from *a* and *b*. It has two dimensions, even though it is a line. This use of "dimension" is analogous to the modern algebraic term "degree." One is reminded at this point of the traditional doctrine that there is no comparison between magnitudes different in kind. Thus, one cannot add a line to a surface, nor can one say by how much the one exceeds the other. Descartes, however, relaxes this claim in problems in which unity is determined. It may be noted that division, being the converse of multiplication, lowers the dimension of the quotient. Thus, *ab/c* is of the first dimension. Descartes does not say what taking roots does to dimensionality. The reader may, perhaps, be able to determine this for himself.

[7] Descartes is discussing the technique for reducing several equations in several unknowns to a single equation in one unknown. The second part of the paragraph describes the resulting equation by employing the convention that unknown lines are expressed by letters at the end of the alphabet, '*z*' for instance, while 'known' or 'given' lines are represented by letters at the beginning of the alphabet ('*a*' through '*d*' in the text). Note that all the examples he gives have terms of the same dimension in a single equation.

square, or the supersolid, or the square of the cube, etc. is equal to something produced by the addition or subtraction of two or more other quantities, of which one is known and the others are composed of some mean proportionals between the unit and this square, or cube, or square of the square, etc., multiplied by other knowns. I write this in this manner:

$z = b$, or

$z^2 = -az + bb$, or

$z^3 = +az^2 + bbz - c^3$, or

$z^4 = az^3 - c^3z + d^4$, etc.

This is to say, *z*, which I take for the unknown quantity, is equal to *b*, or the square of *z* is equal to the square of *b* less *a* multiplied by *z*. Or the cube of *z* is equal to *a* multiplied by the square of *z* plus the square of *b* multiplied by *z* less the cube of *c*. And so with the others.

And we can always so reduce all the unknown quantities to one alone, when the problem can be constructed by circles and straight lines, or also by conic sections, or even by another line which is only one or two degrees more composed.[8] But I do not stop to explain this in more detail because I would take away from you the pleasure of learning it of yourself and the utility of cultivating your mind by exerting it, which, in my opinion, is the principal thing that we can draw from this science. Also because I note nothing so difficult that those who are a little versed in common geometry and in algebra, and who attend to all that is in this treatise, cannot find.

And so I content myself here with warning you that, provided that in disentangling these equations we do not fail to make use of all the divisions which are possible, we will infallibly have the most simple terms to which the question can be reduced.

What are the plane problems.

And if it can be resolved by ordinary geometry, that is to say, by using only straight and circular lines traced on a plane surface, when the last equation shall have been entirely disentangled, all that will remain at most is one unknown square, equal to something produced by the addition or subtraction of its root multiplied by some known quantity and of some other quantity also known.

How they are resolved.

And then this root, or unknown line, is easily found. For if I have, for

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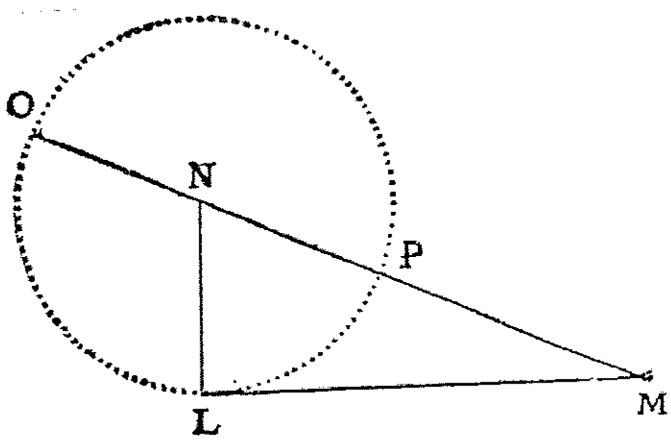
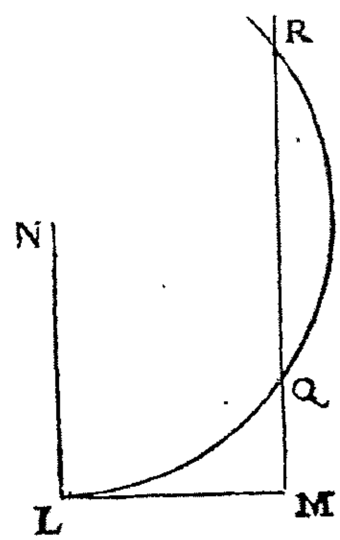
[8] The term 'degree' is consistently used by Descartes to modify curves, not terms. At present he has not defined the degree of a curve. It will turn out that conic sections are first degree curves. Descartes does not justify this claim here. Moreover, it is a bit puzzling, in that he has just above associated constructability in general with the possibility of reducing the several equations in several unknowns to a single equation in one unknown. Now he seems to be limiting the constructions to those carried out by certain curves.

example, $z^2 = az + bb$, I make the right triangle NLM, of which the side LM is equal to b, the square root of the known quantity bb, and the other LN is $\frac{1}{2}a$, the half of the other known quantity, which has been multiplied by z, which I suppose to be the unknown line. Then prolonging MN, the hypotenuse of this triangle, up to O, so that NO is equal to NL, the whole OM is z the line sought. And it is expressed in this way, $z = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$.[9]



But if I have $yy = -ay + bb$, and y is the quantity that must be found, I make the same right triangle NLM, and from its hypotenuse MN I take away NP equal to NL, and the rest PM is y the root sought, in such a way that I have $y = -\frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$. And quite the same, if I had $x^4 = -ax^2 + b^2$, then PM would be x^2. And I should have $x = \sqrt{-\frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}}$; and so with the others.[10]

Finally if I have $z^2 = az - bb$, I make NL equal to $\frac{1}{2}a$, and LM equal to b as before. Then in place of joining the points MN, I draw MQR parallel to LN. And from the center N through L having described a circle which cuts it in the points Q & R, the line sought z is MQ or else MR, for in this case it is expressed in two manners, to wit



[9] Descartes is assuming that a and b are known lines, z the unknown. It is somewhat surprising that Descartes does not make use of his definitions of the operations in solving the equation here. The reader must satisfy himself that OM is a magnitude such that OM x OM = 2NL x OM + LM x LM. It is a separate claim that OM $= \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2}$, and this claim, too, stands in need of demonstration. For a more methodical procedure than that of Descartes, see Exercises, p. 84.

[10] Note here that in $x^4 = -ax^2 + b^2$ the equation is composed of terms of different dimension. This must mean that the problems which led to them differed in the way described above on pp. 48-49. In the present case, the problem which led to this equation must have been a problem in which unity was determined. For some relevant exercises, see Exercises, p. 85.

$$z = \frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb}\text{ , \& } z = \frac{1}{2}a - \sqrt{\frac{1}{4}aa - bb}\,.$$

And if the circle, which has its center at the point N, passing through the point L, neither cuts nor touches the straight line MQR, there is no root in the equation, so that we can be sure that the construction of the problem proposed is impossible.

Moreover these same roots can be found by an infinity of other means, and I have only wished to put these [here] as very simple in order to show that we can construct all the problems of ordinary geometry, without doing anything but the little that is included in the four figures which I have explained. This is something I do not believe the ancients have noted, for otherwise they would not have taken the trouble of writing so many books where the order alone of their propositions lets us know that they did not have the true method for finding them all, but that they only amassed what they encountered.

Example drawn from Pappus.

And we can also see this quite clearly from what Pappus has put at the beginning of his seventh book, where, after having spent some time to name all that had been written in geometry by those who had preceded him, he speaks finally of a question which he says neither Euclid, nor Apollonius, nor any other had been able to resolve entirely. And here are his words:

I cite the Latin version rather than the Greek text so that everyone will understand it more easily.

> 'Now [Apollonius] says in the third book that the locus [problem][11] to three and four lines had not been completed by Euclid, nor was he, nor anyone else, able to complete it; but neither was he able to add any little bit to those things which Euclid wrote through merely those conics which had been demonstrated up to Euclid's time.'

And a little after he explains thus what this question is.

> 'But the locus to three and four lines, in which [Apollonius] pompously boasts and shows off, no thanks given to him who had written before, is such. If, three straight lines given in position, straight lines be drawn from one and the same point in given

[11] Proclus, a Greek commentator on Euclid, defines locus and locus theorem as follows: "I call locus-theorems those which deal with the same property throughout the whole of a locus, and a locus I call a position of a line or surface which has throughout one and the same property." For examples of, and exercises with locus problems, see Exercises, pp. 86-87.

angles to the three lines, and the proportion be given of the rectangle contained by two lines drawn to the square on the remaining, the point falls on a solid locus given in position, that is, one of the three conic sections.[12] And if to four straight lines given in position lines are drawn in given angles, and the proportion be given of the rectangle contained by two lines drawn to that contained by the remaining two, the point similarly falls on a section of a cone given in position. So if only to two the locus has been shown [to be] plane.[13] But if to more than four, the point falls on loci not yet known, but only called lines; what sort they are or what property they have is not established: one of these, not the first, and which seems most manifest, they have constructed showing it to be useful. Now their propositions are these.

'If from some point to five straight lines given in position straight lines be drawn in given angles and the proportion is given of the rectangular parallelepipidal solid that is contained by three lines drawn to the rectangular parallelepipidal solid that is contained by the remaining two and any given line, the point falls on a line given in position. But if to six and the proportion is given of the solid contained by three lines to the solid that is contained by the remaining three, again the point falls on a line given in position. But if to more than six, they are no longer able to say whether the proportion is given of anything contained by four lines to what is contained by the remaining, because there is nothing contained by more than three dimensions.'

Here I ask you to note in passing that the scruple that the ancients had made of using the terms of arithmetic in geometry (who could not progress from the fact that they could not see the relation of them very clearly) caused much obscurity and perplexity in the manner in which they were expressed. For Pappus continues in this manner.

[12] The meaning which Pappus gives to the term, "given" is probably the same as that which Euclid defines in the *Data*:

1. Areas, lines and angles are said to be given in magnitude when we can make others equal to them.
2. A ratio is said to be given when we can make another equal to it.
3. Rectilinear figures are said to be given in species when their angles are severally given and the ratios of the sides one towards another are also given.
4. Points, lines and angles are said to be given in position when they always occupy the same place.
5. A circle is said to be given in magnitude when the radius is given in magnitude.
6. A circle is said to be given in position and in magnitude when the center is given in position and the radius in magnitude.

[13] For the meaning of the designations "plane" and "solid" loci, see note 24 in book 2.

'But they agreed with those who a little earlier expounded these things, not signifying that what is contained by these as one thing comprehensible in any manner. It will be possible, however, through composed proportions both to express and demonstrate all these things in the stated proportions, and even to them according to this mode. If from some point to straight lines given in position straight lines be drawn in given angles, and there is given the proportion composed from those which the first has to the first, and the second to the second, and the third to the third, and the remaining to a given line, if there are seven (but if eight, and the remaining to the remaining): then the point falls on lines given in position. And similarly however many they are, odd or even in multitude, since these, as I have said, resemble the locus to four lines, they have therefore set forth nothing so that the line be known.'

The question then, which had begun to be solved by Euclid and pursued by Apollonius, without having been completed by anyone, is such. Having three or four or a greater number of straight lines given in position, first one requires a point, from which one can draw as many other straight lines, one for each of the givens, which make with them given angles, and that the rectangle contained by two of those, which are so drawn from the same point, have the given proportion with the square of the third, if there are only three of them; or else with the rectangle of the two others, if there are four of them; or else, if there are five of them, that the parallelepipidal solid composed of three have the given proportion with the parallelepipidal solid composed of the two that remain and another given line. Where there are six, that the parallelepipidal solid composed of three has the given proportion with the parallelepipidal solid of the three others. Where there are seven, that that which is produced when we multiply four of them, one by the other, has the given ratio with that which is produced by the multiplication of the three others, and yet another given line. Where there are eight, that the product of the multiplication of four has the given proportion with the product of the other four. And thus this question can be extended to every other number of lines. Then, because there is always an infinity of diverse points which can satisfy what has here been required, it is also required to know and to trace the line on which they must all be found. And Pappus says that when there are only three or four given straight lines, it is one of the three conic sections. But he undertakes neither to determine nor to describe it. Nor does he elucidate those [lines] where all the points must be found, when the question is proposed in a greater number of lines. He only adds that the ancients had imagined one of them, which they showed to be useful, but which would seem the most manifest, and which yet would not be the first. This is what has given me occasion to attempt if by the method which I make use of we can go yet farther than they had.

Response to the question of Pappus.

And first I have learned that this question, being proposed in only three or four or five lines, we can always find the points sought by simple geometry, that is to say, by using only the rule and compass, nor doing anything else than what has already been stated, except only when there are five given lines, if these are all parallel. In which case, as also when the question is proposed in six, or seven, or eight, or nine lines, we can always find the points sought by the geometry of solids, that is to say, by employing some one of the three conic sections. Except only when there are nine given lines, if they are all parallel. In which case once more, and again in ten, eleven, twelve, or thirteen lines, we can find the points sought by means of a curved line which would be of one degree more composed than the conic sections. Except in thirteen if they are all parallel, in which case, and in fourteen, fifteen, sixteen, and seventeen, it is necessary to employ a curved line yet one degree more composed than the preceding. And so to infinity.[14]

Then I also found that when there are only three or four given lines, the points sought are all found, not only on one of the three conic sections, but sometimes also on the circumference of a circle, or on a straight line. And that when there are five, or six, or seven, or eight of them, all these points are found on any one of the lines which are of a degree more composed than the conic sections, and it is impossible to imagine any among them [i.e., the conic sections] which would not be useful for the question; [15] but they can also once again be found on a conic section, or on a circle, or on a straight line. And if there are nine, or ten, or eleven, or twelve of them, these points are found on a line which can only be one degree more composed than the preceding; but all those which are one degree more composed can be used, and so on to infinity.

Finally, the first, and the most simple of all after the conic sections is that which can be described by the intersection of a parabola and a straight line, in

[14] Note that Descartes, having distinguished in the previous paragraph finding one point which answers to the conditions set in the locus from discovering and constructing the line containing all such points, in this paragraph is speaking only of the first. The present text is a promise only—its justification is given at the end of the first book of *The Geometry*. In the next paragraph, Descartes is speaking of the second half of the locus problem, the discovery and construction of the locus itself, i.e., of the line containing the points that meet the conditions.

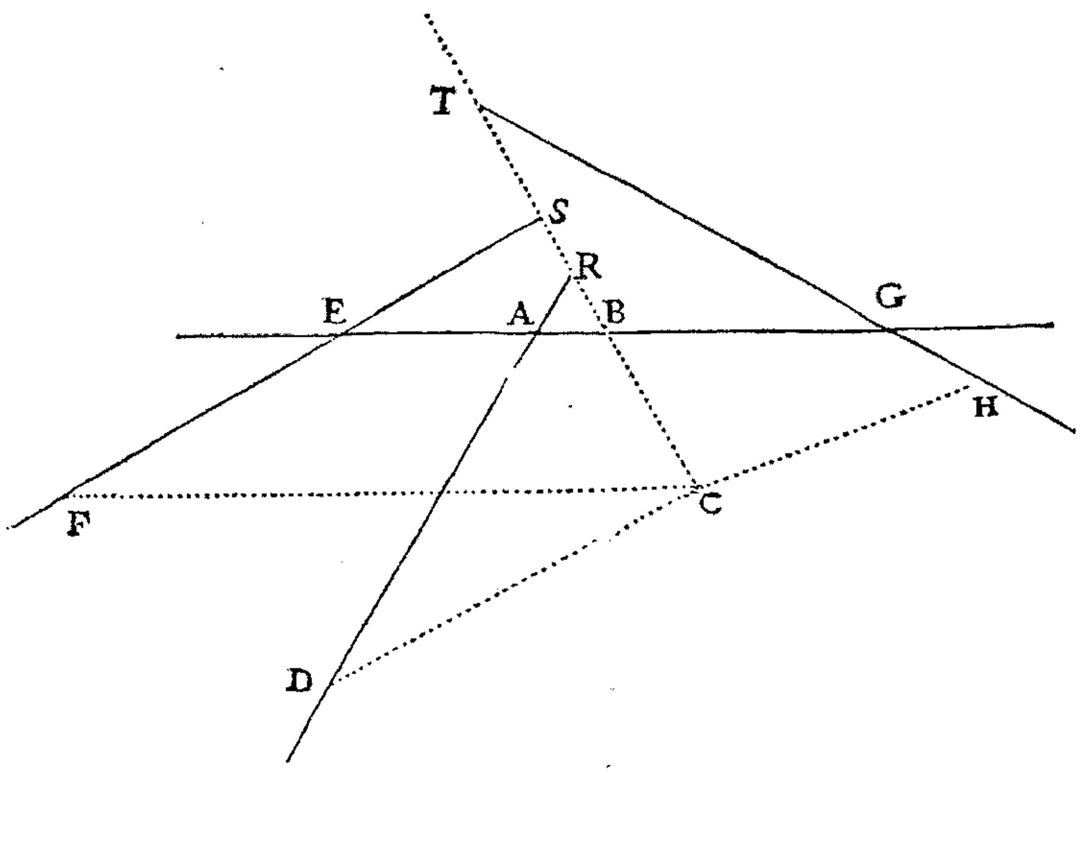
[15] It seems that Descartes is asserting that curves of this class—this degree—will always be the solutions to the locus problem of the corresponding number of lines. That is, he is asserting that the locus problem of 5-8 lines is commensurately universal with curves one degree higher, or "more composed," than the conics, except for those 5-8 line problems which yield straight lines, circles, or conic sections. If this is true, the investigation of the n-line locus problem is the investigation of all curves. This should be kept in mind in reading the introductory matter in the second book, whose subtitle reads, "On the nature of curved lines."

the manner which will soon be explained. In this way I believe I have entirely satisfied what Pappus tells us was sought by the ancients. And I will try to put the demonstration in a few words. For I am tired already from writing so much.

Let AB, AD, EF, GH, etc. be many lines given in position and let it be necessary to find a point, such as C, from which, having drawn other straight lines to the given ones, such as CB, CD, CF, and CH, such that the angles CBA, CDA, CFE, CHG, etc. are given, and that what is produced by the multiplication of one part of these lines be equal to what is produced by the multiplication of the others, or else that they have some other given proportion, for this does not make the question more difficult.[16]

How we must set the terms to come to the equation in this example.

First I suppose the thing as already done, and to disentangle myself from the confusion of all these lines, I consider one of the givens and one of those which must be found, for example AB and CB, as the principal ones and to

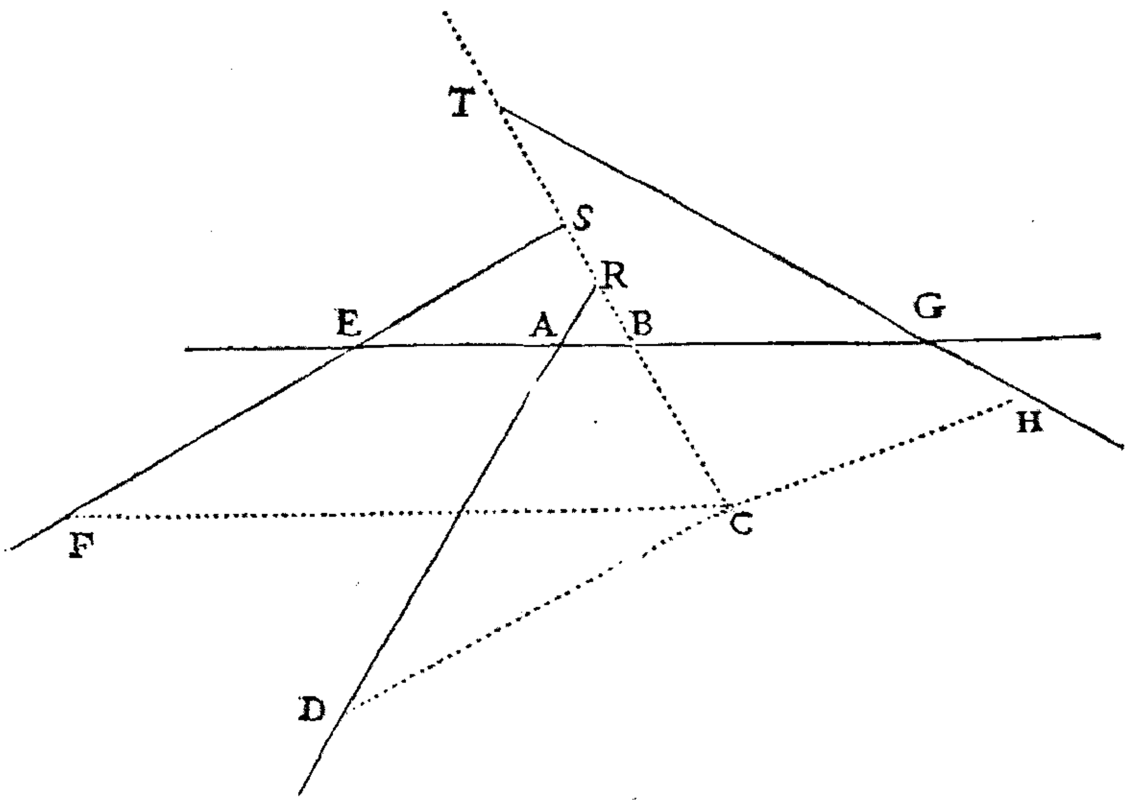


[16] The figure above, which is used throughout the argument in this book and again in the second book, uses solid lines for given and dotted lines for unknown parts of the figure. C is supposed to be a point on the locus, and it is placed in a particular spot only to assist the imagination. There is nothing in the problem as stated to prevent C's being, for example, within the angle RAB. Moreover, since C is merely one of the points on the locus, there may be C's in other regions of the plane or even some in all. When, in what follows, Descartes discusses the various signs that may connect the terms in the expressions for CD, CF, CH, he seems to have in mind various initial placements of the point C. The student may wish to sketch some of these possibilities to achieve a firmer grasp on the meaning of the text.

which I shall thus try to relate all the others. Let the segment of the line AB, which is between the points A and B, be named *x*, and let BC be named *y*. And let all the other given lines be lengthened, until they meet these two, also lengthened if there is need, and if they are not parallels to them. So you see here that they cut the line AB at the points A, E, G, and BC at the points R, S, T. Then, because all the angles of the triangle ARB are given, the proportion between the sides AB and BR is also given, and I set it as that of *z* to *b*, in such a manner that, AB being *x*, RB will be $\frac{bx}{z}$, and the whole CR will be $y+\frac{bx}{z}$ because the point B falls between C and R;[17] for if R fell between C and B, CR would be $y-\frac{bx}{z}$ and if C fell between B and R, CR would be $-y+\frac{bx}{z}$. Again, the three angles of the triangle DRC are given, and consequently also the proportion which is between the sides CR and CD, which I set as that of *z* to *c*, in such a manner that CR being $y+\frac{bx}{z}$, CD will be $\frac{cy}{z}+\frac{bcx}{zz}$. After this, since the lines AB, AD, and EF are given in position, the distance which is between the points A and E is also given, and if we name it *k*, we will have EB equal to $k + x$; but this would be $k - x$, if the point B fell between E and A; and $-k + x$, if E fell between A and B. And because the angles of the triangle ESB are given, the proportion of BE to BS is also given, and I set it as *z* to *d*, so that BS is $\frac{dk+dx}{z}$, and the whole CS is $\frac{zy+dk+dx}{z}$; but this would be $\frac{zy-dk-dx}{z}$, if the point S fell between B and C; and this would be $\frac{-zy+dk+dx}{z}$, if C fell between B and S. Further, the three angles of the triangle FSC are given, and therefore the proportion of CS to CF, which would be as *z* to *e*, and the whole CF will be $\frac{ezy+dek+dex}{zz}$. In the same manner AG which I name *l* is given, and BG is $l-x$, and, because of the triangle BGT, the proportion of BG to BT is also given, which would be as *z* to *f*, and BT will be $\frac{fl-fx}{z}$, and CT = $\frac{zy+fl-fx}{z}$. Then again the proportion of TC to CH is given, because of the triangle TCH, and then setting them as *z* to *g*, we will have CH = $\frac{+gzy+fgl-fgx}{zz}$.

And so you see that, in any such number of lines given in position that we may have, all the lines drawn from the point C at given angles according to the tenor of the question can always be expressed, each by three terms; of which

[17] The line *z* is an arbitrary standard length, taken at pleasure. It could even be our unit. Once taken for the first calculation, however, it must remain the same in length.



one is composed of an unknown quantity y, multiplied or divided by another known quantity, and another composed of the unknown quantity x, also multiplied or divided by some other known, and the third of a quantity altogether known.[18] Except only if they are parallel, either to the line AB, in which case the term composed of the quantity x will be nil, or else to the line BC, in which case that which is composed of the quantity y will be nil, both of which are too manifest for me to stop to explain.[19] And as for the signs + and -, which are joined to these terms, they can be changed in every manner imaginable.[20]

Then you also see that, multiplying several of these lines, one by the other, the quantities x and y, which are found in the product, can only have as many dimensions as there are had lines for the explication of which they serve,

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[18] The lines through C are CB, CD, CF, and CH, which lines are involved as factors in the two products whose ratio is given. Descartes here observes that the most complex form they take is ± Ax ± By ± C, where A, B, C stand for the coefficients obtained above. For example, CH is $\frac{gzy + fgl - fgx}{z^2}$, or

$$-\left(\frac{fg}{z^2}\right)x + \left(\frac{g}{z}\right)y + \frac{fgl}{z^2}.$$

[19] The student should be able to show that what Descartes claims here is true. First replace one line, say AD, with a line parallel to AB, and draw CD as before, from C at a fixed angle to the new given line. Prove in this case that the line CD (drawn to the new given line) has no x-term. Next, make the replacement for AD parallel to BC, and prove that there is no y-term.

[20] The student should look back to pages 55-57 to see whether this is strictly true.

which have been thus multiplied, in such a way that they never have more than two dimensions in that [line] which is produced only by the multiplication of two lines, nor more than three in that [line] which is produced only by the multiplication of three, and so *in infinitum*.

How we find that this problem is plane, when it is not proposed in more than five lines.

Further, in order to determine the point C, there is but a single condition which is required, to wit that that which is produced by the multiplication of a certain number of these lines be equal, or (which is no more difficult) has the given proportion, to that which is produced by the multiplication of the others; we can take at our discretion one of these two unknown quantities x or y, and seek the other by this equation. In which it is evident that when the question is not proposed in more than five lines, the quantity x which does not serve for the expression of the first can always have only two dimensions,[21] so that taking a known quantity for y, there will only remain $xx \pm ax \pm bb$.[22] And thus we could find the quantity x with rule and compass, in the manner just now explained. Indeed successively taking infinitely diverse lengths for the line y, we should so find infinitely many of them for the line x, and so we should have an infinity of diverse points, such as that which is marked C, by means of which we could describe the curved line required.

It can be so done when the question is proposed in six or a greater number of lines, if there are among the givens some which are parallel to BA, or to BC, so that one of the two quantities x or y have only two dimensions in the equation and so that one could find the point C with rule and compass. But, on the contrary, if they are all parallel, although the question is proposed in only five lines, the point C cannot be so found, because the quantity x not being found in any equation, it will no longer be allowed to take a known quantity for what is named y, but this will be what one must seek,; and because what has three dimensions, we cannot find but by drawing the root of a cubic equation, which cannot be generally done without employing at the least a conic section.

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[21] What Descartes says here is true only with qualification. We need to add that this first x-free line is multiplied by two other C-lines, and not by the other, given, line. Otherwise the x-term will have three dimensions.

[22] Descartes has now added the last part of the given, namely that the product of two C-lines shall equal or have a given ratio to the product of the remaining two. If we have CD x CB = CF x CH, for example, we will have, upon substitution: $y\left(\frac{cy}{z}+\frac{bcx}{z^2}\right)=\left(\frac{ezy+dek+dex}{z^2}\right)\left(\frac{gzy+fgl-fgx}{z^2}\right)$. Considering only the kind of terms (that is, collecting as new known, or given, quantities all those composed of given quantities), we have: $\pm Ay^2 \pm Bxy \pm Cx \pm Dy \pm E = x^2$. (Note that these capital letters indicate terms of different dimension. Thus A and B have no dimensions, C and D one, and E two.) This is the equation to which our locus problems have been reduced. Now, since it is an indeterminate problem, i.e., since it has new x's for each y, we may solve it only by taking a single y and finding the determinate x (might there be more than one?) that go with that y. This means that y has become a given quantity, so that our equation is now $x^2 = \pm\, ax \pm b^2$, as Descartes concludes.

And although there are up to nine given lines, provided that they are not all parallel, we can always make it that the equation rises only to the square of the square, by means of which we can thus always resolve it by the conic sections, in the manner which I will explain hereafter. And although there are up to thirteen, we can always make it that it rises only to the square of the cube. It follows that we can resolve it by means of a line that is only one degree more composed than the conic sections, in the manner that I shall also explain hereafter.[23] And this is the first part of what I have to demonstrate here, but before I pass to the second it is necessary that I say something in general about the nature of curved lines.

[23] See Exercises, p. 87, for a schematization of Descartes's conclusion.

Second Book

Concerning the nature of curved lines.

What are the curved lines that we can receive in geometry.

The ancients noted quite well that among the problems of geometry, some are plane, others are solid, and other are linear, that is to say that some can be constructed by tracing only straight lines and circles, while other cannot be unless we employ at least some conic section, and yet others, unless we employ some other line more composed.[24] But I am astonished that they did not distinguish, beyond these, diverse degrees among the lines more composed, and I cannot comprehend why they named them mechanical rather than geometrical. For to say that this was because it was necessary to make use of some machine to describe them, it would be necessary to reject by the same argument circles and straight lines, seeing that we cannot describe them on paper but with a compass and a rule, which we can also call machines. This is not because the instruments which serve to trace them, being more composed than the rule and compass, cannot be as exact; for it would be necessary for this reason to reject them from mechanics, where the exactness of the works which proceed from the hand is desired, rather than from geometry, where it is only the exactness of the reasoning that we seek, and which can no doubt be as perfect concerning these lines as concerning the others. I also will not say that this is because they did not wish to increase the number of their postulates and that they were content that we grant them that they can join two given points by a straight line and describe a circle with a given center which passes through a given point. For they have not made any scruple about supposing, beyond these, in order to trace the conic sections, that we can cut a given cone by a given plane. And it is not necessary to suppose anything, in order to trace the curved lines that I claim to introduce here, except that two or more lines can be moved, one upon the other,[25] and their intersections mark others, which seems

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[24] Pappus divides problems into plane, solid, and linear as follows:

> The ancients considered three classes of geometric problems, which they called plane, solid, and linear. Those which can be solved by means of straight lines and circumferences of circles are called plane problems, since the lines or curves by which they are solved have their origin in a plane. But problems whose solutions are obtained by the use of one or more of the conic sections are called solid problems, for the surfaces of solid figures [i.e., the conic surfaces] have to be used. There remains a third class which is called linear because other 'lines' than those I have just described, having diverse and more involved origins, are required for their construction. Such lines are the spirals, the quadratrix, the conchoid, and the cissoid, all of which have many important properties.
>
> Pappus, vol. 1, p. 55, prop. 5, book 3.

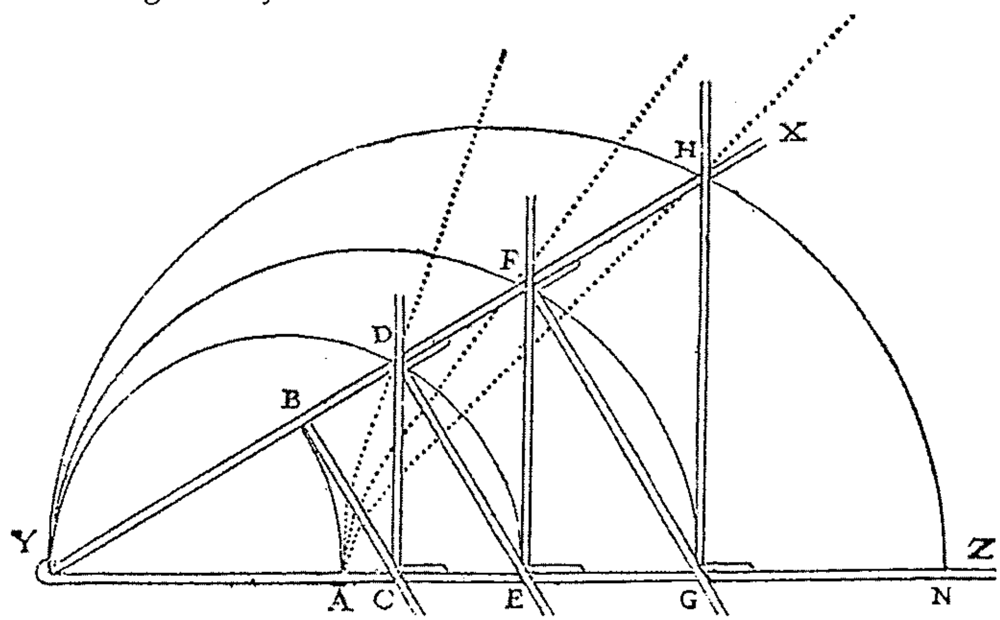
[25] The French for "one upon the other" is "l'une par l'autre," which could also be rendered "one by another," or "one through another."

to me no more difficult. It is true that they did not quite entirely receive the conic sections into their geometry, and I do not undertake to change the names which have been approved by usage, but it is, it seems to me, very clear, that taking as one does for geometric what is precise and exact, and for mechanical what is not so, and considering geometry as a science which teaches generally how to know the measure of all bodies, we must no more exclude the more composed lines than the more simple, provided that we can imagine them described by a continuous movement, or by many [movements] which follow one another and of which the latter are entirely ruled by those which precede. For by this means we can always have an exact knowledge of their measure. But perhaps what prevented the ancient geometers from receiving those which are more composed than the conic sections is that the first which they considered had by chance been the spiral, the quadratrix, and the like, which truly pertain only to mechanics and are not among the number of those which I think should here be received, because we imagine them described by two separate movements, which do not have between them any relation which can be exactly measured. Although they had later examined the conchoid, the cissoid, and a few others which are among them,[26] still because they did not perhaps sufficiently note their properties, they took no more account of these than of the first. Or else it is that, seeing that they yet knew only a few things concerning the conic sections and there even remained for them much concerning what can be done with rule and compass that they were ignorant of, they believed they ought not enter into matter more difficult. But, since I hope that from now on those who have the cleverness to make use of the geometric calculation here proposed will not find much to be held up by concerning the plane or solid problems, I believe it to the purpose that I invite them to other researches where they will never lack exercise.

Consider the lines AB, AD, AF, and the like which I suppose to have been described by the aid of the instrument YZ,. which is composed of many rulers so joined that, the one which is marked YZ being fixed to the line AN, we can open and close the angle XYZ and, when it is completely closed, the points B, C, D, F, G, H are all gathered at the point A; but so that, in the measure in which we open it, the ruler BC, which is joined at right angles with XY at the point B, pushes toward Z the ruler CD, which runs along YZ, while always making right angles with it, and CD pushes DE, which runs just the same along YX, while staying parallel to BC, DE pushes EF, EF pushes FG, which pushes GH. And we can conceive an infinity of others, which are pushed consecutively in the same manner and of which some always make the same angles with YX and the others with YZ. Now, while we open the angle XYZ, the point B describes the line AB, which is a circle, and the other points D, F, H, where the intersections of the other rulers are made, describe other curved lines

[26] On the spiral, quadratrix, cissoid, and conchoids, see Exercises, pp. 87-90.

AD, AF, AH, of which the latter are in order more composed than the first and this than the circle. But I do not see what can prevent our conceiving as clearly and as distinctly the description of this first as of the circle, or at least as of the conic sections, nor what can prevent our conceiving the second and the third and all the others that we can describe as well as the first, nor consequently what can prevent our receiving them all in the same way to serve in the speculations of geometry.



The manner of distinguishing all the curved lines in certain genera. And of knowing the relation which all their points have to those of straight lines.

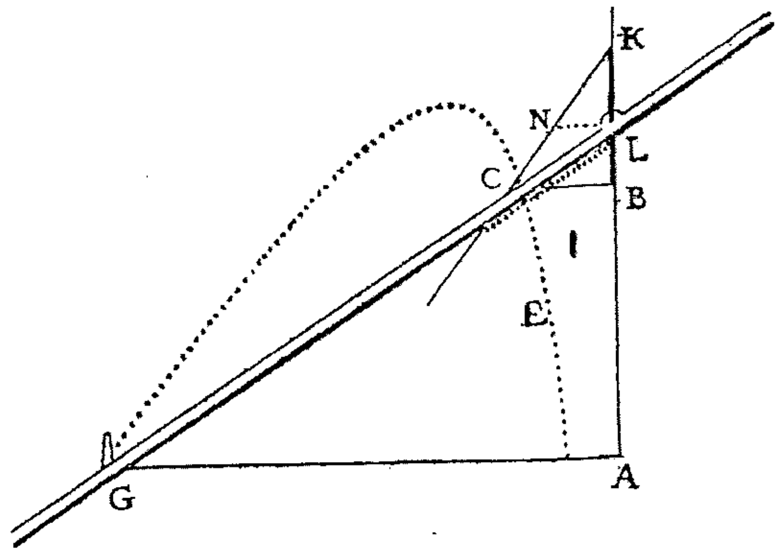
I could put here many other ways to trace and conceive of curved lines, which would be more and more composed by degrees to infinity. But to understand together all those that are in nature [*en la nature*] and to distinguish them in order in certain genera, I know nothing better than to say that all the points of those curves which we can call geometric, that is to say, which fall under some precise and exact measure, necessarily have some relation to all the points of one straight line, which can be expressed by some equation, all of them by one and the same.[27] And when this equation rises only up to a

[27] The equations for these three curves may be obtained as follows (adapted from the Dover edition, p. 47, note 75):

1) Let YA = YB = a, YC = x, CD = y, YD = z. Then $z : x :: x : a$, so that $z = \frac{x^2}{a}$. But $z^2 = x^2 + y^2$, so AD's equation is $x^4 = a^2 (x^2 + y^2)$.

2) Let YA = YB = a, YE = x, EF = y, YF = z. Then $z : x :: x :$ YD, so that YD = $\frac{x^2}{z}$. Also, x : YD :: YD : YC, whence YC = $\frac{\frac{x^4}{z^2}}{x} = \frac{x^3}{z^2}$. But YD : YC :: YC : a, and therefore $\frac{ax^2}{z} = \left(\frac{x^3}{z^2}\right)^2$, or

rectangle of two indeterminate quantities, or else to the square of the same one, the curved line is of the first and most simple genus, in which only the circle, the parabola, the hyperbola, and the ellipse are included. But when the equation rises up to the third or fourth dimension of two or of one of the two indeterminate quantities (for two of them are needed to explain here the relation of one point to another), it is of the second, and when the equation rises up to the fifth or sixth dimension, it is of the third and so with the others to infinity.



As, if I wish to know what the genus is of the line EC, which I imagine to be described by the intersection of the ruler GL and of the rectilinear plane figure CNKL, the side of which, KN, is indefinitely lengthened toward C, and which plane figure, being moved along the plane underneath in a straight line (that is to say, in such a way that its diameter KL is always found to be applied upon some part of the line BA lengthened in one direction and the other), moves the ruler GL circularly around the point G, because it is so joined that it passes always through the point L. I choose a straight line, such as AB, to relate to its diverse points all those of this curved line EC, and on this line AB I choose a point, such as A, to begin this calculation with it. I say that I choose both the one and the other, because one is free to take such as he will. For, although there are many choices which would make the equation shorter and easier, nonetheless in whatever manner we take them, we can always make the line appear of the same genus, as is easy to demonstrate. After this, taking a point on the curve at my discretion, as C, upon which I suppose the instrument which serves to describe it is applied, I draw from the point C the line CB parallel to GA and, because CB and BA are two undetermined and unknown quantities, I name the one y and the other x. But to find the relation of the one to the other, I also consider the known quantities which determine the description of this curved line, such as GA, which I name a, KL, which I name b, and NL parallel to

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$z = \sqrt[3]{\frac{x^4}{a}}$. But again $z^2 = x^2 + y^2$, so AF's equation is $\sqrt[3]{\frac{x^8}{a^2}}$, or $x^8 = a^2(x^2 + y^2)^3$.

3) In the same way it can be shown that the equation of AH is $x^{12} = a^2(x^2 + y^2)^5$.

For exercises with this hyper-compass and its relevance to the classical problem of doubling the cube, see Exercises, p. 91.

GA, which I name c. Then I say, as NL is to LK, or c to b, so is CB, or y, to BK, which is consequently $\frac{b}{c}y$, and BL is $\frac{b}{c}y-b$, and AL is $x+\frac{b}{c}y-b$. Further, as CB is to LB, or y is to $\frac{b}{c}y-b$, so is a, or GA, is to LA, or $x+\frac{b}{c}y-b$, so that multiplying the second by the third we produce $\frac{ab}{c}y-ab$, which is equal to $xy+\frac{b}{c}yy-by$, which is produced by multiplying the first by the last. And thus the equation which must be found is $yy=cy-\frac{cx}{b}y+ay-ac$, from which we know that the line EC is of the first genus, as in fact it is nothing other than an hyperbola.[28]

If in the instrument which serves to describe the curve, in place of the straight line CNK, there were this hyperbola, or any other curved line of the first genus, which lies in the plane CNKL, the intersection of this line and of the ruler GL will describe, in place of the hyperbola EC, another curved line, which will be of the second genus. As if CNK is a circle, of which L were the center, we will describe the first conchoid of the ancients;[29] and if this is a parabola of which the diameter were KB, we will describe the curved line which I have just

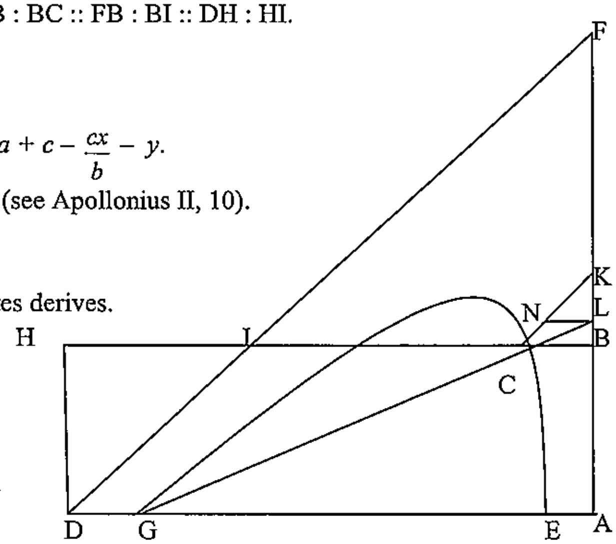
[28] [The following proof is adapted from footnote 86 of the Dover edition.] Extend AG to D so that DG = EA. Since E is a point on the curve where GL coincides with GA, then EA = NL = c, and thus, DG = NL. Draw DF parallel to KC, and therefore cutting AB extended at F.

Now, let GCE be an hyperbola with DF and AF as asymptotes (see Apollonius, *Conic Sections*, II, 4 and 8). We now must prove that this is the curve described by the machine that Descartes is using.

Extend BC through the curve to DF, cutting DF at I, and then draw DH parallel to AF and cutting IB extended at H. Therefore KL : LN :: KB : BC :: FB : BI :: DH : HI.

But DH = AB = x.
Thus, $b : c :: x$: HI.
Whence, HI = $\frac{cx}{b}$, and IB = $a + c - \frac{cx}{b}$, and IC = $a + c - \frac{cx}{b} - y$.
But in any hyperbola, rect. IC, BC = rect. DE, EA (see Apollonius II, 10).
Whence, $y(a + c - \frac{cx}{b} - y) = ac$,
OR, $y^2 = cy - \frac{cxy}{b} + ay - ac$, the equation Descartes derives.

What happens when the triangle KNL has reached the position where K and F coincide? What happens when K is above F? Can one prove that FD and FA are asymptotes for this hyperbola? For additional exercises with Descartes's machine, see Exercises, p. 91.



[29] Replacing the triangle with the semicircle will manifestly describe the conchoid defined above, if the semicircle is drawn to the left of AB. What curve is described by the semicircle to the right of AB? Can other conchoids be derived from this machine?

now said to be the first and the most simple for the question of Pappus, while there are only five straight lines given in position.[30] But if in place of these curved lines of the first genus, there is one of those of the second, which lies in the plane CNKL, we will describe by its means one of those of the third; or if there is one of those of the third, we will describe one of those of the fourth, and so on to infinity, as it is very easy to know by the calculation. And in any other manner that we can imagine the description of a curved line, provided that it be among the number of those which I call geometric, we can always find an equation to determine all its points in this way.

Furthermore, I put the curved lines which make this equation rise up to the square of the square, in the same genus with those which make it rise only up to the cube. And those whose equation rises to the square of the cube [I put] in the same genus with those of which it rises only to the supersolid, and so with the others. The reason for which is that there is a general rule for reducing to cubes all the difficulties which go to the square of the square, and to the supersolid all those which go to the square of the cube, so that we must not consider them more composed.[31]

But it remains to remark that, among the lines of each genus, although the greater part would be equally composed in such a way that they can serve to determine the same points and to construct the same problems, still there are also some that are more simple and that are not as vast in their power. As among those of the first genus, beside the ellipse, the hyperbola, and the parabola, which are equally composed, the circle is also included, which manifestly is more simple, and among those of the second class there is the common conchoid, which has its origin in the circle, and there are still some others which, although they are not as vast [in their power] as most of those of the same genus, cannot all the same be put in the first [genus].

There follows the explication of the question of Pappus posited in the preceding book.

Now, after having so reduced all the curved lines to certain genera, it is easy for me to pursue the demonstration of the response which I just made to the question of Pappus. For first having shown above that when there are only three or four given straight lines, the equation which serves to determine the

[30] The reader has no way of knowing, at this point, whether Descartes is justified in making the claim that the curve drawn with a parabola sliding along the vertical is the solution to a 5-line locus. The justification is made in a part of *The Geometry* that we will not read, but begins on p. 83 of the Dover edition.

[31] Descartes does not provide the rule for reduction until Book III (which we will not be reading in this tutorial). The reader is being asked to trust him at this point. What had been obscure in the doctrine of the genera of curves, namely, why there are two dimensions for every degree, is here made manifest. The reader may wish to consider the implications for the notion "genus" when there are infinitely many genera of curves, and when their difference are numerical and concern the techniques of solution of equations. The conic sections, it may be noted, all fall in the same genus according to the Cartesian schema—but *not* because they are the sections of a cone.

points sought does not rise but up to the square, it is evident that the curved line where these points are found is necessarily some one of those of the first genus: because this same equation explains the relation which all the points of the lines of the first genus have to those of a straight line. And [it is evident] that when there are no more than eight given straight lines, this equation rises only to the square of the square at most, and that consequently the line sought can only be of the second genus, or lower. And [it is evident] that when there are no more than twelve given lines, the equation rises only up to the square of the cube, and consequently the line sought is only of the third genus or lower. And so with the others. And because the position of the given straight lines can vary in every way, and consequently make both the known quantities and the equation's signs + and - change in every way imaginable, it is evident that there is not any curved line of the first genus which would not be useful to this question, when it is proposed in four straight lines, nor any of the second which would not be useful, when it is proposed in eight, nor of the third, when it is proposed in twelve, and so with the others. So that there is not any curved line which falls under calculation and can be received in geometry which would not be useful for some number of lines.[32]

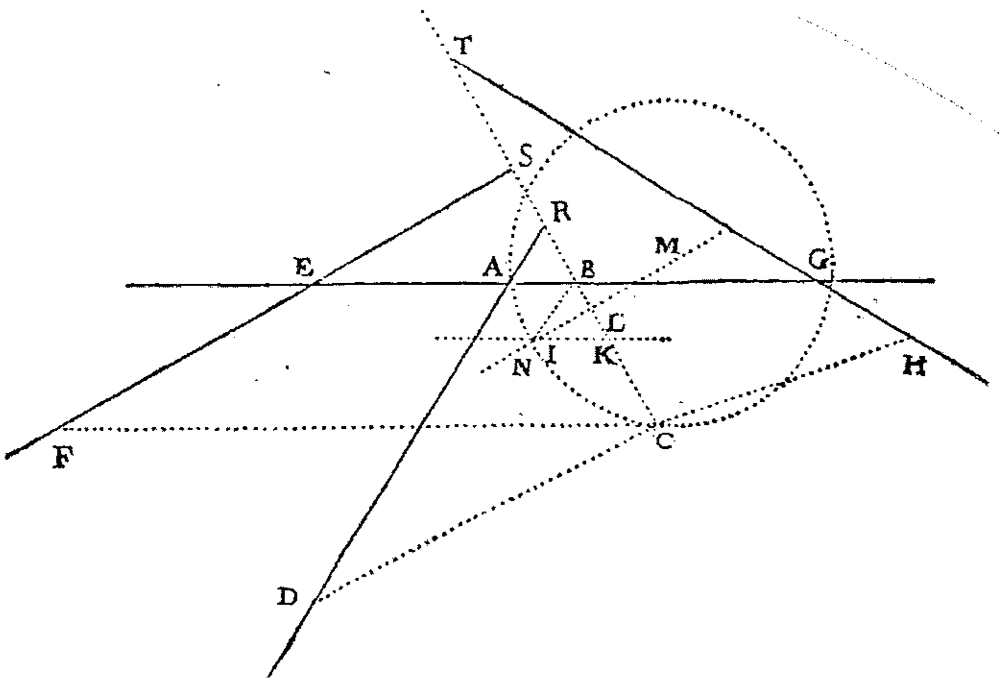
Solution of this question when it is proposed only in three or four lines.

But it is more particularly necessary here that I determine and give the manner of finding the line sought, which [manner] serves in each case when there are only three or four given straight lines, and we will see by the same means that the first genus of curved lines contains no others than the three conic sections and the circle.

Let us again take the four lines AB, AD, EF, and GH, given above, and let it be necessary to find another line on which are found an infinity of points, such as C, from which, having drawn the four lines CB, CD, CF, and CH at given angles to the given lines, CB multiplied by CF produces a sum equal to CD multiplied by CH,[33] that is to say, having made CB = y, CD = $\frac{czy + bcx}{zz}$, CF

[32] If the above is true, then the locus problem in its full generality is the problem whose solution is all geometric curves—that is, by definition, all curves reducible to equations in two unknowns. At this point one may ask, why not start with an equation and only two reference lines, and examine the problem from there? The given lines in the locus, having produced the equation, serve no further purpose in the investigation. That Descartes proceeds to ignore these given lines (with the notable exception of AB) may be seen in what follows.

[33] Notice that Descartes has taken a particular pair of lines, CB and CF, to have the same product as another pair, CD and CH. Let us consider the kinds of terms found in the resulting equation. In the first place, they are all of one dimension. Second, some have x alone, some y alone, and one, namely, $-\left(\frac{dez^2 + cfgz - bcgz}{ez^3 - cgz^2}\right)xy$, has both x and y. None, however, has neither x nor y. This will have unhappy consequences below. By way of anticipation, let it be noted that a term formed entirely of given quantities would have been generated had Descartes taken CB x CD = CF x CH, as the reader may have noticed in



$= \frac{ezy + dek + dex}{zz}$, and CH = $\frac{gzy + fgl - fgx}{zz}$, the equation is:

$$yy = \frac{(-dekzz + cfglz)y + (-dezzx - cfgzx + bcgzx)y + bcfglx - bcfgxx}{ezzz - cgzz},$$

at least when supposing *ez* greater than *cg*.[34] For if it were less, it would be necessary to change all the + and - signs. And if the quantity *y* be found nil, or less than nothing in this equation, while the point C has been supposed in the angle DAG, it would be necessary to suppose it in the angle DAE, or EAR, or RAG, while changing the signs + and - according to what would be required to this effect.[35] And if the value of *y* be found nil in all these four positions, the

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note 20, above.

[34] With the figure as drawn, *ez* is necessarily greater than *cg*, as may be seen from the following argument:

CR > CB and CT > CS

Hence	CR x CT > CB x CS
And also	CR x CT x CF > CB x CS x CF
But, since	CB x CF = CD x CH
Then	CR x CT x CF > CD x CH x CS
Hence	CR : CD > CH x CS : CT x CF
But	CR : CD :: *z* : *c* [See p. 57.]
	CH : CT :: *g* : *z*
	CS : CF :: *z* : *e*
Therefore	*z* : *c* > *g* : *e*, and so *ez* > *cg* Q.E.D.

[35] Descartes seems to have in mind the possibility that, no matter what *x* is, the quadratic equation in *y*, when solved, would equate *y* with a quantity to be subtracted without anything for it to be subtracted from. This can only result from an equation of the type $y^2 = -ay - b^2$ — the only type that was omitted on page 51. This suggests that Descartes is making some use of "quantities less than nothing," at least in

question would be impossible in the case proposed. But let us suppose the one here to be possible, and to abridge the terms, in place of the quantity $\frac{cfglz - dekzz}{ez^3 - cgzz}$ let us write $2m$, and in place of $\frac{dezz + cfgz - bcgz}{ez^3 - cgzz}$ let us write $\frac{2n}{z}$, and thus we have $yy = 2my - \frac{2n}{z}xy + \frac{bcfglx - bcfgxx}{ez^3 - cgzz}$, of which the root is:

$$y = m - \frac{nx}{z} + \sqrt{mm - \frac{2mnx}{z} + \frac{nnxx}{zz} + \frac{bcfglx - bcfgxx}{ez^3 - cgzz}}\,.$$

And to abridge again, in place of $-\frac{2mn}{z} + \frac{bcfgl}{ez^3 - cgzz}$, let us write o, and in place of $\frac{nn}{zz} - \frac{bcfg}{ez^3 - cgzz}$, let us write $\frac{p}{m}$. For these quantities being all given, we can name them as we please. And so we have:

$$y = m - \frac{n}{z}x + \sqrt{mm + ox - \frac{p}{m}xx}\,,$$

which must be the length of the line BC, leaving AB, or x, undetermined.[36] And

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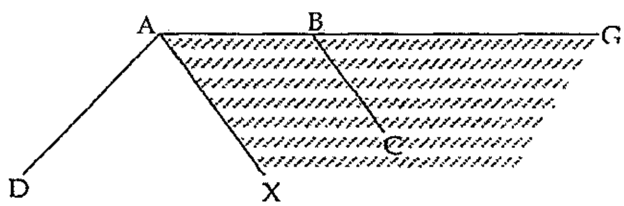
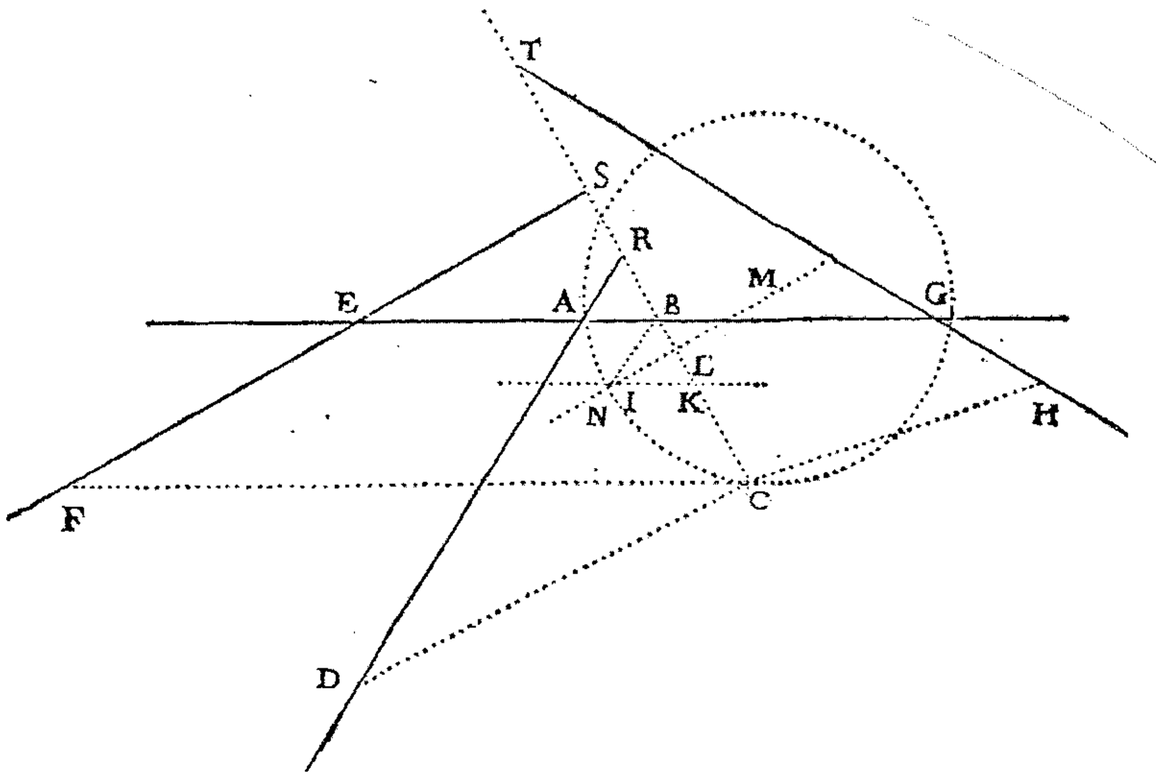
investigation, if not in results. The use, however, is to develop an absurdity. His conclusion is that C was impossibly located (as within angle DAG) and that the analysis needs to begin again at the start. The change of signs referred to in the text seems to apply to the signs attached to the terms expressing CD, CF, CH and those lines used in finding them (CR, CS, CT). He seems to have erred, however, in dividing the plane into four regions by the lines AD and AB, since if C is to the left of the parallel to CB drawn through A, certain signs will also have to be changed. The equation as given at the top of this page will only result if C is located within the angle BAX.



[36] In solving the equation with the simplified coefficients, Descartes presumes that he has an equation like $y^2 = ay + b^2$ (p. 51). He might, however, have one like $y^2 = ay - b^2$ (also p. 51), and therefore should at least consider the possibility that there are two points C that would be on the locus. Thus the last equation on the page would more suitably read $y = m - \frac{n}{z}x \pm \sqrt{m^2 + ox + \frac{p}{m}x^2}$.

Moreover, there is nothing which necessitates that px^2/m and ox have + signs attached to them, provided that o, p, and m signify lines, as Descartes has specified above (p. 48). (Indeed, in the original French edition px^2/m here and on p. 71 is negative, and so it has been translated here, but the algebra indicates that it should be positive.) What about m^2, the x-free term under the radical? It always arises from squaring $\frac{2m - \frac{2nx}{z}}{2}$ according to the paradigmatic solutions of the quadratic equations given on pp. 51-52. But here the consequences of taking CB x CF and CD x CH, rather than CB x CD and CF x CH, become manifest. If Descartes had chosen the latter approach, then in addition to the m^2 term there would have also been another second dimension term without x, say q^2, whose sign of combination might have been either + or -. In this case Descartes would have had to make one more substitution, letting $M^2 = m^2 \pm q^2$. The co-efficient of x^2, namely p/m, should then have become p/M. Thus, that the first term inside the radical, m^2, is the square of the first term outside is not generally the case, but depends on having taken the terms CB, CD, CF, CH in a particular way.

it is evident that when the question is proposed only in three or four lines, we can always have such terms, except that some of them can be nil, and the signs + and - can be changed in diverse ways.[37]



After this, I make KI equal and parallel to BA, in such a way that it cuts from BC the part BK equal to *m*, because here it is +*m*;[38] and I would have added it by drawing the line IK on the other side, if it had been -*m*; and I would not have drawn it at all, if the quantity *m* had been nil. Then I also draw IL so that the line IK is to KL as *z* is to *n*, that is to say, so that IK being *x*, KL is $\frac{n}{z}x$. And by the same means I also know the proportion which is between KL and

[37] At this point, Descartes has merely simplified and solved the equation to which the locus problem had been reduced. This spells out in greater detail what had been asserted above at the end of book 1, on pp. 57-60. Commenting on this part of *The Geometry*, Descartes says in a letter to Mersenne: "In regard to the problem of Pappus, I have given only the construction and demonstration without putting in all the analysis; . . . in other words, I have given the construction as architects build structures, giving the specification and leaving the actual manual labor to carpenters and masons" (*Oeuvres* [Paris, 1824], vol. 7, p. 157; translation from Dover edition of *The Geometry*).

[38] Here Descartes begins to construct the line which is the locus, taking first certain simple, special, cases of the radical. He sees that one part of BC, or *y*, is simpler than the other, and constructs that first part, $m-\frac{nx}{z}$, in this paragraph, and the more complex part in the next one.

IL, which I put as that between n and a, that is, KL being $\frac{n}{z}x$, IL is $\frac{a}{z}x$.[39] And I take the point K as between L and C because here there is $-\frac{n}{z}x$; instead I would have put L between K and C if I had had $+\frac{n}{z}x$; and I would not have drawn this line IL if $\frac{n}{z}x$ had been nil.

Now, this done, there remains for me no more for the line LC than these terms, LC = $\sqrt{mm + ox - \frac{p}{m}xx}$, from which I see that if they were nil, the point C would be found on the straight line IL, and that if they were such that their root can be drawn, that is to say, such that, mm and $\frac{p}{m}xx$ being marked with the same sign + [or -],[40] oo would be equal to $4pm$, or else that the terms mm and ox, or ox and $\frac{p}{m}xx$ were nil, this point C would be found on another straight line which would be no more difficult to find than IL.[41] But when these do not occur, the point C is always on one of the three conic sections, or on a circle, of which one of the diameters is on the line IL, and the line LC is one of the lines which are applied ordinatewise to this diameter, or on the contrary, LC is parallel to the diameter to which the line IL is applied ordinatewise. To wit, if the term $\frac{p}{m}xx$ is nil, this conic section is a parabola, and if it is marked with the sign +, it is an hyperbola, and finally, if it is marked with the sign – , it is an ellipse. Except only if the quantity aam is equal to pzz and that the angle ILC is right, in which case we have a circle in place of an ellipse. If this section is a parabola, its upright side is equal to $\frac{oz}{a}$, and its diameter is always on the line

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[39] Perhaps the best way to understand Descartes's introduction of a is to note that the triangle ILK is given in species; hence the ratio of any pair of its sides is also given. Now, n is a given quantity. Hence a, which is to be a fourth proportional to KL and IL, will also be given. When IK equals z, LK will be n, and IL, a.

One can easily prove that if any length l on AB, extended both ways, be cut off by a pair of lines parallel to BC, the corresponding length cut off by those same lines on IL, extended if necessary, will be $\frac{a}{z} \times l$. Whence, $a : z$, or $\frac{a}{z}$, may be thought of as the ratio of transfer which adjusts the length of any line on AB to the corresponding line on IL, and conversely, $z : a$ is the ratio of transfer from IL to AB.

[40] The Latin edition of 1683 mentions only the adding symbol, but the original French edition of 1637 and all subsequent French editions insert the subtraction symbol. Which edition is correct?

[41] On these possibilities, see Exercises, p. 91.

IL.[42] And to find the point N, which is its vertex, it is necessary to make IN equal to $\frac{amm}{oz}$; and let the point I be between L and N, if the terms are + *mm* + *ox*; but let the point L be between I and N, if they are + *mm* - *ox*; or else N must be between I and L, if there is - *mm* + *ox*. But one can never have [merely] - *mm*, in the way that the terms have been put here. And finally, the point N would be the same as the point I if the quantity *mm* were nil. By means of which it is easy to find this parabola by the first problem of the first book of Apollonius.[43]

If the line asked for is a circle or an ellipse or an hyperbola,[44] one must first seek the point M, which is its center, and which is always on the straight line IL, where we find it by taking $\frac{aom}{2pz}$ for IM. So that if the quantity *o* is nil, this center is exactly at the point I. And if the line sought is a circle, or an ellipse, we must take the point M on the same side as the point L, with respect to the point I, when we have $+ox$, and when we have $-ox$, we must take it on the other. But quite the contrary in the hyperbola, if we have $-ox$, this center M must be toward L, and if we have $+ox$, it must be on the other side. After this,

[42] This may be seen as follows:

Letting *u* stand for the upright side of the section at N, then by the nature of the parabola, $LC^2 = LN, u$. [See Apollonius I, 11.]

But LN = IL + IN, if we take the first case (noted in the next sentence), where I is between N and L. (Does the same conclusion arise if we take any of the other cases?)

Also, let ϕ stand for IN.

Then, since IL = ax/z, we have LN = $ax/z + \phi$.

But the equation Descartes says belongs to the parabola is $LC^2 = m^2 + ox$.

Therefore, $axu/z + \phi u = m^2 + ox$.

Equating coefficients, we have $au/z = o$, and thus, $u = oz/a$, as Descartes says.

Also by equating coefficients, $\phi u = m^2$.

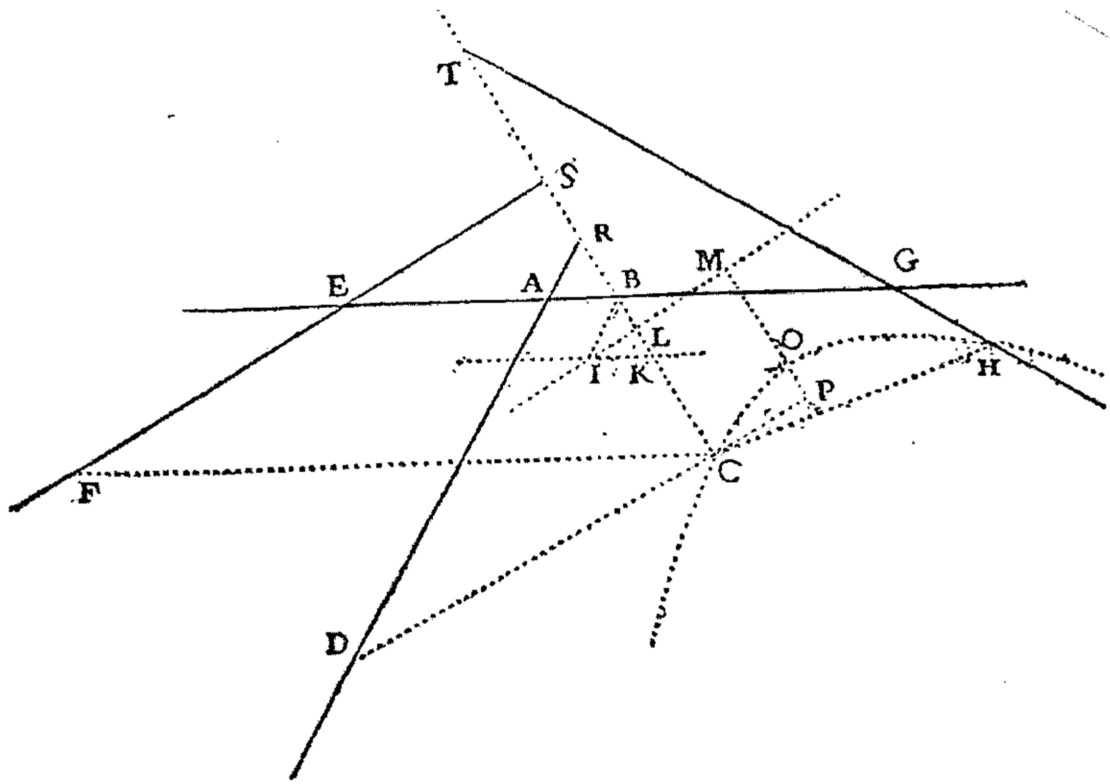
Whence, $\phi oz/a = m^2$, and $\phi = am^2/oz$, as Descartes says in the following sentence.

(On the procedure of equating coefficients, see Exercises, p. 92.)

[43] The enunciation of the first problem in Apollonius *On Conic Sections* I (proposition 52) is as follows: "Given a straight line bounded at one point, to find in the plane the section of a cone called 'parabola,' whose diameter is the given straight line, and whose vertex is the end of the straight line, and where whatever straight line is dropped from the section to the diameter at a given angle, will equal in square the rectangle contained by the straight line cut off by it from the vertex of the section and by some other given straight line." The argument of this proposition considers only the right angle, whereas proposition 53 considers the case of any other angle, so it may be thought of as a continuation of the same problem. Descartes appears to be thinking this way in his ordinal numbering below.

[44] In order to prove the things Descartes asserts in the following sentences, one may follow a strategy similar to (but somewhat more complicated than) that used for the parabola in footnote 42, this time appealing to *On Conic Sections* I, 12 and 13. After equating coefficients, one will now derive three equations—one for each part of LC under the radical—which will allow one to construct the distance to the center (IM), from which we can then construct for the upright side, the transverse side, and the distance to the vertex (IN).

the upright side of the figure must be $\sqrt{\frac{oozz}{aa}+\frac{4mpzz}{aa}}$, when we have + mm and the line sought is a circle or an ellipse, or else when we have $-mm$ and it is an hyperbola. And it must be $\sqrt{\frac{oozz}{aa}-\frac{4mpzz}{aa}}$, if, the line sought being a circle or an ellipse, we have $- mm$, or else if, it being an hyperbola and the quantity oo being greater than $4mp$, we have $+mm$. If the quantity mm is nil, this upright side is $\frac{oz}{a}$, and if ox is nil, it is $\sqrt{\frac{4mpzz}{aa}}$. Then, for the transverse side, one must find a line, which would be to this upright side, as aam is to pzz, to wit, if this upright side is $\sqrt{\frac{oozz}{aa}+\frac{4mpzz}{aa}}$, the transverse is $\sqrt{\frac{aaoomm}{ppzz}+\frac{4aam^3}{pzz}}$. In all these cases the diameter of the section is on the line IM, and LC is one of those lines which are applied to it ordinatewise. And so, making MN equal to half the transverse side and taking it on the same side of the point M as is the point L, we have the



point N for the vertex of this diameter. Following which it is easy to find the section by the second and third problems of the first book of Apollonius.[45]

But when, this section being an hyperbola, we have + mm and the quantity oo is nil or smaller than $4pm$, we must draw from the center M the line

[45] Presumably Descartes means Apollonius, *On Conic Sections* I, 54-59.

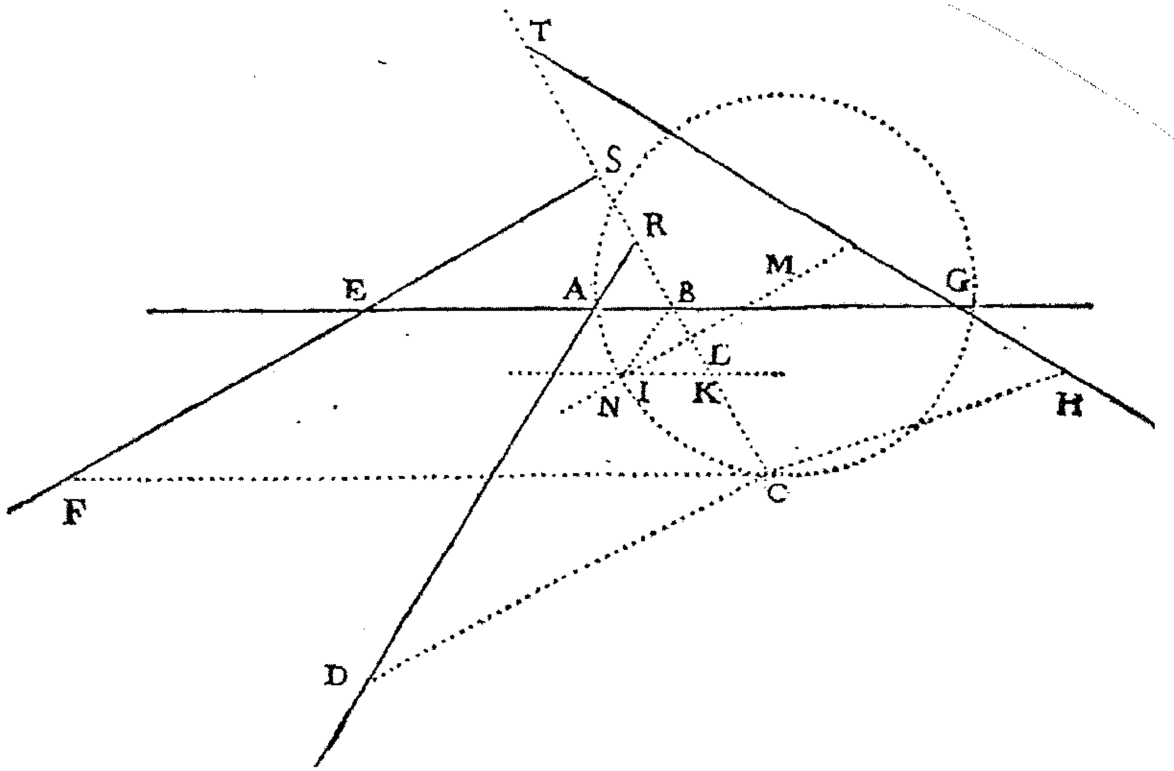
MOP parallel to LC, and CP parallel to LM, and make MO equal to $\sqrt{mm - \frac{oom}{4p}}$; or else make it equal to m, if the quantity ox is nil. Then, we must consider the point O as the vertex of this hyperbola, whose diameter is OP, and CP the line which is applied to it ordinatewise, and its upright side is $\sqrt{\frac{4a^4m^4}{ppz^4} - \frac{a^4oom^3}{p^3z^4}}$ and its transverse side is $\sqrt{4mm - \frac{oom}{p}}$. Except when ox is nil, for then the upright side is $\frac{2aamm}{pzz}$, and the transverse is $2m$. And so it is easy to find it by the third problem of the first book of Apollonius.[46]

The Demonstration of all that has just been explained.

And the demonstrations of all this are evident. For composing a space of the quantities which I have assigned for the upright side and the transverse, and for the segment of the diameter NL, or OP, following the tenor of the eleventh, twelfth, and thirteenth theorems of the first book of Apollonius, we will find all the same terms, of which the square of the line CP, or CL, which is applied ordinatewise to this diameter, is composed. As in this example, subtracting IM, which is $\frac{aom}{2pz}$, from NM, which is $\frac{am}{2pz}\sqrt{oo + 4mp}$, I have IN. Adding IL, which is $\frac{a}{z}x$, to this, I have NL, which is $\frac{a}{z}x - \frac{aom}{2pz} + \frac{am}{2pz}\sqrt{oo+4mp}$, and this being multiplied by $\frac{z}{a}\sqrt{oo+4mp}$, which is the upright side of the figure, one gets $x\sqrt{oo+4mp} - \frac{om}{2p}\sqrt{oo+4mp} + \frac{moo}{2p} + 2mm$ for the rectangle, from which a space must be subtracted which would be to the square of NL as the upright is to the transverse. And the square of NL is: $\frac{aa}{zz}xx - \frac{aaom}{pzz}x + \frac{aam}{pzz}x\sqrt{oo+4mp} + \frac{aaoomm}{2ppzz} + \frac{aam^3}{pzz} - \frac{aaomm}{2ppzz}\sqrt{oo+4mp}$, which must be divided by aam and multiplied by pzz, because these terms express the proportion between the transverse side and the upright, and it comes to $\frac{p}{m}xx - ox + x\sqrt{oo+4mp} + \frac{oom}{2p} - \frac{om}{2p}\sqrt{oo+4mp} + mm$. This must be subtracted from the preceding rectangle, and we find $mm + ox - \frac{p}{m}xx$ for the square of CL, which consequently is a line applied ordinatewise in an ellipse or in a circle to

[46] Apollonius, *On Conic Sections* I, proposition 60.

the segment of the diameter NL.



And if we wish to explain all the given quantities by numbers, take, for example, EA = 3, AG = 5, AB = BR, BS $=\frac{1}{2}$ BE, GB = BT, CD = $\frac{3}{2}$ CR, CF = 2CS, CH = $\frac{2}{3}$ CT, and let the angle ABR be 60 degrees, and finally, let the rectangle contained by CB and CF be equal to the rectangle contained by CD and CH; for one must have all these things so that the question is entirely determined. And with these things, supposing AB = x, and CB = y, we find in the manner explained above that $yy = 2y - xy + 5x - xx$, and $y = 1 - \frac{1}{2}x + \sqrt{1 + 4x - \frac{3}{4}xx}$; so that BK must be 1, and KL must be half of KI, and, since the angle IKL, or ABR, is 60 degrees, and KIL, which is half of KIB or IKL, is 30 degrees, ILK is right.[47] And since IK or AB is named x, KL is $\frac{1}{2}x$, and IL is $x\sqrt{\frac{3}{4}}$, and the quantity which was above named z is 1, that which was a is $\sqrt{\frac{3}{4}}$, that which was m is 1,

[47] In order to argue that angle KIL is 30 degrees, Descartes appears to be assuming that angle KIL is half angle KIB, which means he is assuming that IK = KB, and therefore that $x = 1$. Is it fair to make this assumption? Can angle KIL being 30 degrees be proved without assuming $x = 1$?

that which was o is 4, and that which was p is $\frac{3}{4}$, in such a way that we have $\sqrt{\frac{16}{3}}$ for IM and $\sqrt{\frac{19}{3}}$ for NM, and since *aam*, which is $\frac{3}{4}$, is here equal to *pzz*, and the angle ILC is right, we find that the curved line NC is a circle. And we can easily examine all the other cases in the same manner.

What are the plane and solid loci and the manner of finding them.

Because the equations which only rise to the square are all included in what I have just explained, not only the problems of the ancients in 3 and 4 lines is entirely solved here, but also all that which pertains to what they called the composition of solid loci, and consequently to the plane loci, since they are included in the solid ones. For these loci are nothing else than, when it is a question of finding some point for which one condition is lacking to be entirely determined, as occurs in this example, all the points of one and the same line can be taken for what is sought. And if this line is straight, or circular, we call it a plane locus. But if it is a parabola or an hyperbola, or an ellipse, we call it a solid locus. And whenever it is thus, we can come to an equation which contains two unknown quantities, and it is parallel to any of those which I have resolved. But if the line which so determines the point sought is of a degree more composed than the conic sections, we can call it, in the same manner, a supersolid locus, and so with the others.[48] And if two conditions are lacking for the determination of this point, the locus where it is found is a surface, which can be plane or spherical or more composed. But the highest end which the ancients ever had in this matter was to succeed in the composition of solid loci, and it seems that all that Apollonius wrote on conic sections was in order to seek it.

Further we see here that what I have taken for the first genus of curved lines includes no others than the circle, the parabola, the hyperbola, and the ellipse, which is all that I have undertaken to prove.

. . . .

I shall not stop to consider in detail the lines corresponding to the other cases, for I have not undertaken to speak of this; and, having explained the method of finding an infinity of points passing through any of them, I think I have given a way to describe them.

Which are the curved lines that we can describe by finding many of their points, which can be received in geometry.

[48] The third and last book of *The Geometry* is entitled, "On the construction of problems that are solid or supersolid."

It is also to the purpose to remark that there is a great difference between this manner of finding many points to trace a curved line and that which one makes use of for the spiral and its like. For in the latter we do not find indifferently all the points that we seek, but only those that can be determined by some measure more simple than that which is required to compose it, and so, to speak properly, we do not find one of its points. That is to say, not one of those which are so proper to it that they can only be found by it. Whereas, there is no point in the lines serving for the question proposed that cannot be found among those that are determined in the manner just now explained. And because this manner of tracing a curved line by indifferently finding many of its points extends only to those that can also be described by a regular and continuous movement, we must not entirely reject it from geometry.

Which are those that we describe with a string, which can be received.

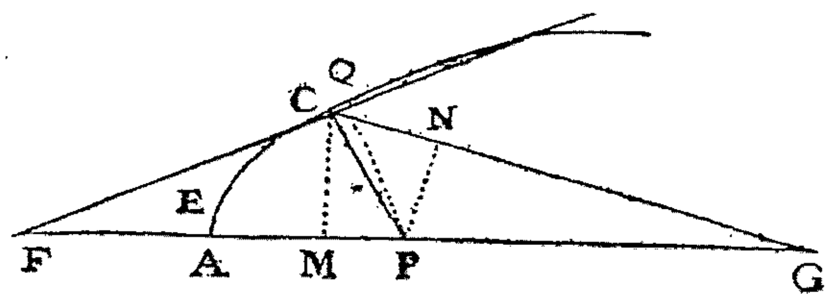
And we must not reject that in which we make use of a thread or a looped string to determine the equality or the difference of two or more straight lines that can be drawn from each point of the curve that we are seeking to certain other points, or to certain other lines at certain angles. So we have done in the *Dioptrics,* to explain the ellipse and the hyperbola. For, while we cannot receive any lines that seem like to strings, that is to say, that are now straight, now curved, because the proportion that exists between the straight and the curved is not known and even, I believe, cannot be known by men, we cannot conclude anything about it that would be exact and assured. Yet, because we make use of strings in these constructions only to determined straight lines, whose length we know perfectly, this must not make us reject them.

To find all the properties of curved lines, it suffices to know the relation that all their points have to those of straight lines, and the manner of drawing other lines which cut them at those points at right angles.

Now, from this alone, that we know the relation which all the points of a curved line have to all those of a straight line, in the manner that I have explained, it is easy to find as well the relation which they have to all other given points and lines, and then to know the diameters, axes, centers, and other lines or points to which each curved line would have some more particular or more simple relation than to the others, and so to imagine diverse ways to describe them and to choose the easiest of them. And we can also, by this alone, find as it were all that can be determined concerning the size of the area which they contain, without there being need that I give more than an opening. And finally, concerning all the other properties that we can attribute to curved lines, they depend only upon the size of the angles that they make with certain other lines. But when we can draw straight lines which cut them at right angles at points where they are met by those with which they make the angles that we wish to measure, or, and I take this to mean the same, which cut their tangents, the size of these angles is no more difficult to find, than those contained between the two straight lines. This is why I believe that I have put here all required for the elements of curved lines, when I have generally given the manner of drawing straight lines, which fall at right angles upon such of their points as we would choose. And I dare say that this is the most useful and most general problem not only that I know, but even that I have ever desired to know, in geometry.

A general method for finding the straight lines that cut given curves, or their tangents, at right angles.

Let CE be the curved line, and let it be required to draw a straight line through point C, which will make with it [the curve CE] right angles. I suppose that the thing has already been done, and that the line sought is CP, which I prolong to point P where it intersects the straight line GA, which I suppose to be that to which one relates all the points of the line CE: in such a way that, making MA or CB = y, and CM or BA = x, I have some equation which explains the relation which is between x and y. And I make PC = s, and PA = v, or PM = v - y, and, because of the right triangle PMC,[49] I have ss, which is the square of the hypotenuse



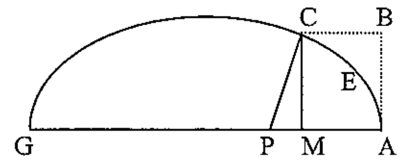
equal to $xx + vv - 2vy + yy$, which are the squares of the two sides. That is to say, I have $x = \sqrt{ss - vv + 2vy - yy}$, or indeed, $y = v - \sqrt{s^2 - x^2}$.[50] And by means of this equation, I remove from the other equation, which explains for me the relation which all the points of the curve CE have to those of the straight line GA, one of the two indeterminate quantities x or y, which is easy to do by putting $\sqrt{s^2 - v^2 + 2vy - y^2}$ everywhere in place of x, and the square of this sum in place of xx, and its cube in place of x^3, and so with the others, if it is x which I wish to remove; or else, if it is y, by putting in its place $v - \sqrt{s^2 - x^2}$, and the square or the cube, etc., of this sum, in place of yy or y^3, etc. In such a way, there remains always after this an equation in which there is but one indeterminate quantity, x or y.

As, if CE is an ellipse, and MA be the segment of its diameter, to which

[49] Descartes appears to have assumed from the outset that the angle AMC is right.

[50] Both here and six lines later the French gives the equations as $y = v + \sqrt{s^2 - x^2}$, which appears to be a typo. There are cases, however, when this quantity could be a sum; in the case of the ellipse above, for example, take C closer to G than to A.

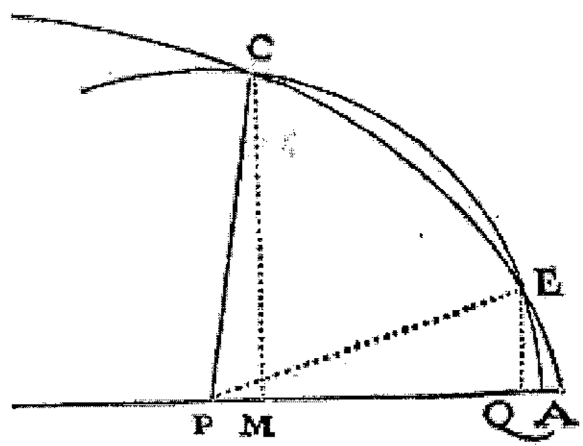
CM must be applied ordinatewise, and which has r for its upright side, and q for the transverse, one has, by the 13th [proposition] of the first book of Apollonius: $xx = ry - \frac{r}{q}yy$, from which, removing xx, there remains $ss - vv + 2vy - yy = ry - \frac{r}{q}yy$, or else, $yy + \frac{qry - 2qvy + qvv - qss}{q - r}$, equal to nothing.



For it is better in this place to consider thus together the whole sum, than to make one part equal to the other.

[Descartes then goes on to give two other examples, which have been omitted.]

Now, after one has found such an equation, instead of using it to know the quantities x or y or z, which are already given, since the point C is given, one must employ it to find v or s, which determine the point P, which is demanded. To this end, it is necessary to consider that if the point P is such as one has desires, the circle of which it makes the center, and which will pass through point C, will there touch the curved line CE, without cutting it; but that if the point P is, however little, closer to or further from the point A than it ought to be, this circle will cut the curve, not only at point C but necessarily also in some other. And one must also consider that, while this circle cuts the curved line CE, the equation by which one seeks the quantity x or y or some similar one, supposing PA and PC to be known, necessarily contains two roots, which are unequal.[51] For if, e.g., this circle cuts the curve at points C and E, having drawn EQ parallel to CM, the names of the indeterminate quantities x and y will just as well fit the lines EQ and QA, as CM and MA; since PE is equal to PC, because of the circle, if, rather than seeking the lines EQ and QA by PE and PA, which one supposes as givens, one will have the same equation as if one should seek CM and MA by PC and PA. Whence it evidently follows that the value of x or of y or of another such quantity which one might have supposed, will be double in this equation, that is to say, that there will be two roots unequal to each other,



[51] By "roots" Descartes does not exactly mean solutions to an equation that is raised to some power. In book three he explains that by "diverse roots" for an equation "I mean to say, values of the [unknown] quantity" (pp. 158-159, Dover edition).

and of which the one will be CM the other EQ, if this is y, and so of the others. It is true that if the point E is not found on the same side of the curve as the point C, only one of these roots would be true, and the other will be reversed, or less than nothing; but the more the points C and E are close to each other, the less will there be a difference between the two roots, and finally they will be entirely equal, if they are two points in one; that is to say, if the circle, which passes through C, touches the curve CE there without cutting it.

Moreover, it is necessary to consider that when there are two equal roots in one equation, it necessarily has the same form as if one multiplies by itself the quantity which one supposes to be unknown in it less the known quantity which is equal to it; and [it is necessary to consider] that after this, if this last form has not so many dimensions as the preceding one, one multiplies by another form which has in it so many as it lacks, to the end that one might have separately an equation between each of the terms of the one, and each of the terms of the other.

As, for example, I say that the first equation found here above, namely, $yy + \frac{qry - 2qvy + qv^2 - qs^2}{q - r}$, ought to have the same form as that which is produced in making e equal to y and multiplying y - e by itself, whence there comes $yy - 2ey + ee$, in such a way that we can compare separately each of their terms,[52] and can say that since the first, which is yy, is entirely the same in the one as in the other, the second which is in the one, $\frac{qry - 2qvy}{q - r}$, is equal to the second of the other, which is $-2ey$, whence, seeking the quantity v, which is the line PA, one has $v = e - \frac{r}{q}e + \frac{1}{2}r$; or else, because we have supposed e equal to y, one has $v = y - \frac{r}{q}y + \frac{1}{2}r$. And so also one would be able to find s by the third term $ee = \frac{qvv - qss}{q - r}$, but, because the quantity v sufficiently determines the point P, which is the only one which we were seeking, one has no need to go further.

[After applying his method to the two other examples, Descartes says the following.]

Therefore if we take AP equal to the above value of v, all the terms of which are known, and join the point P thus determined to C, this line will cut the curve CE at right angles, which was required. And I see nothing to prevent one from extending this problem in the same manner to all curved lines that fall to some geometrical reckoning.

It should be remarked, touching on the last sum [from the third

[52] Notice that Descartes is again equating coefficients; he comments on this technique below.

example], which was taken at one's discretion for filling up the name of the dimensions of the other sum, since it was lacking, as we just now took $y^4 + fy^3 + ggyy + h^3y + k^4$, that the signs + and - one can suppose as one wish without the line v, or AP, turning out differently, as you can easily see by experience. But if I should stop to demonstrate every theorem I mention, it would require a much larger volume than I wish to write. I want rather to inform you in passing that the invention of supposing two equations to be of the same form for the sake of comparing separately the terms of one to those of the other, and so giving rise to many from one alone, of which you have here an example, will serve for an infinity of other problems, and is not one of the least important feature of the method on which I bear myself.

I shall not give the constructions for describing the tangents and perpendiculars sought in connection with the reckoning just explained, because it is always easy to find them, although one often has need of a little skill for giving short and simple ones.[53]

....

[Descartes concludes the third and final book of *The Geometry* as follows.]

But it is not my purpose to write a large book. I am trying rather to include much in a few words, as will perhaps be inferred from what I have done, if it is considered that, while reducing to a single construction all the problems of the same genus, I have at the same time given a method of transforming them into an infinity of other diverse ones, and thus of solving each in an infinite number of ways; that, furthermore, having constructed all plane problems by the cutting of a circle by a straight line, and all solid problems by the cutting of a circle by a parabola; and finally, all that are but one degree more composed by cutting a circle by a curve but one degree composed than the parabola, it is only necessary to follow the same general method to construct all problems, more and more composed, to infinity. For in the case of a mathematical progression, whenever the first two or three terms are given, it is easy to find the rest. I hope that posterity will judge me kindly, not only as to the things which I have explained, but also as to those which I have intentionally omitted so as to leave to others the pleasure of discovery.

[53] For examples, see Exercises p. 92.

Exercises and Problems for Descartes's *Geometry*

Preface to the original version of the *Descartes Notes*

The notes which follow are written in the hope that by them Descartes's own thought may be made more evident to the student, and that thereby the tutorial may proceed as is customary at the College, by discussion of a common and penetrable text, and that the danger of a discussion too much controlled by a tutor with special and private knowledge may be avoided. Individual tutorials should use them as seems appropriate—it is not necessary that the whole be used or the parts to prove helpful.

My own work in preparing this material makes me conscious of the contributions of three years of tutorials to my own understanding of this text. Little of what is contained in these notes is untouched by the insights of one or another of the College community. It is my expectation that amendments and corrections will continue to flow from that spring.

Thanks are due to all who have helped with these studies, but above all, to the light and source of light Who illuminates all studies, without Whom no good thing comes to be, to Whom be honor, praise, and glory, now and forever.

Richard Ferrier

Cartesian Multiplication: (to accompany pp. 47-48)

Descartes has defined multiplication as the finding of a fourth proportional to unity and the two factors. What will happen to the magnitude of a product ab as the line representing unity is changed in length? Let the two lines representing unity be u_1, and u_2, and let $u_1 : u_2 :: k : 1$. What is the ratio of ab taken with u_1, to ab taken with u_2? That is, calling the first product of a and b ab_1, and the second ab_2, find the ratio $ab_1 : ab_2$.

Problem: Using Descartes' definition of multiplication, prove that, in a proportion of four lines, the product of the means equals the product of the extremes, and conversely.

Notes on Resolving the Plane Problems: (to accompany pp. 50-52)

A more methodical procedure for finding roots than that used by Descartes, one using the definitions of the operations and the techniques of ordinary algebra, would be as follows:

If $z^2 = az + b^2$
Then $z^2 - az = b^2$

And therefore, $z^2 - az + \frac{a^2}{4} = b^2 + \frac{a^2}{4}$

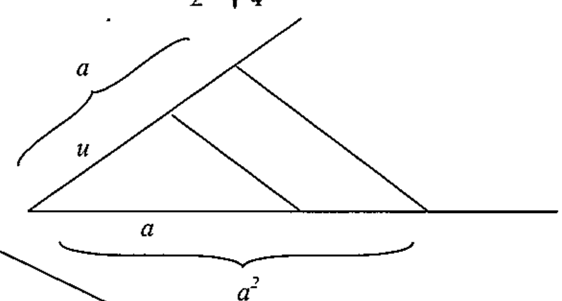
And therefore: $\left(z - \frac{a}{2}\right)^2 = \frac{a^2}{4} + b^2$

$$z - \frac{a}{2} = \sqrt{\frac{a^2}{4} + b^2}$$

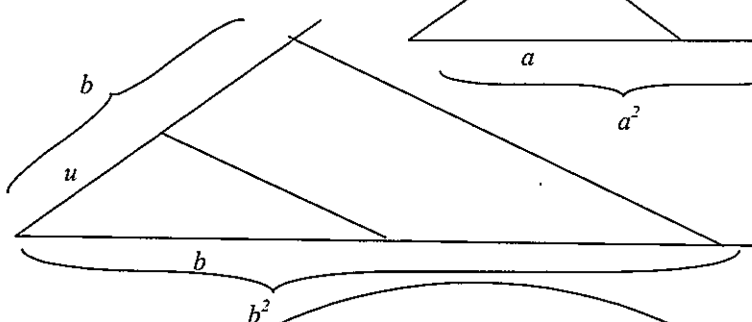
$$z = \frac{a}{2} + \sqrt{\frac{a^2}{4} + b^2}$$

This technique is known as completing the square.

Now, a and b are given lines. Let u be taken arbitrarily as unity. Square a according to the Cartesian definition of multiplication:



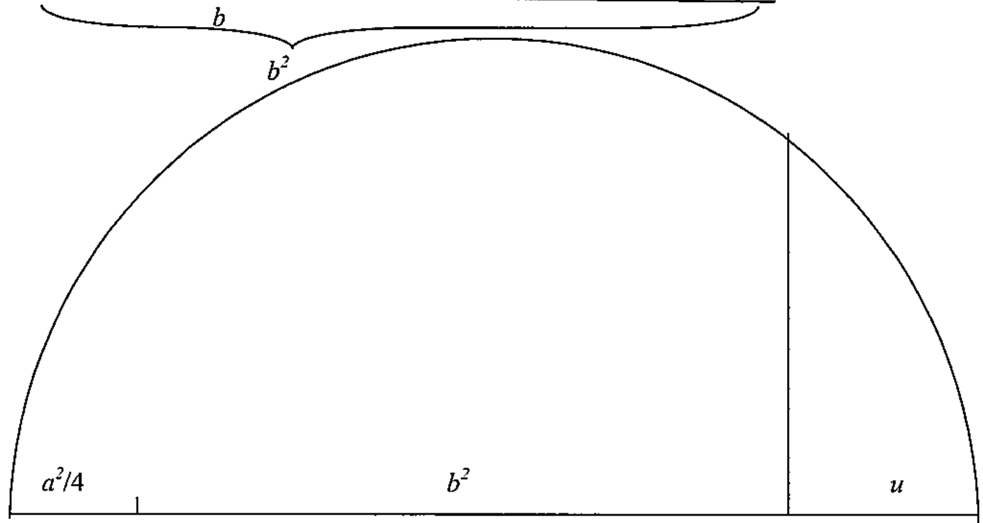
Then divide a^2 by 4:
$a^2/4 =$ _______



Then square b:

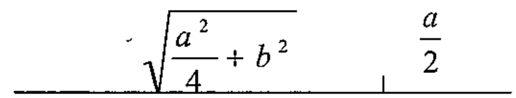
Add $a^2/4$ and b^2, then add unity.

Then construct the semicircle with this sum as base, and erect the perpendicular.



This perpendicular will be, by definition, $\sqrt{\frac{a^2}{4}+b^2}$.

Now add $a/2$.



The resulting line will solve the problem.

Demonstration
Simply reverse the steps in completing the square.

Exercises

1. Construct $z = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2}$ for two different units and compare the two solutions.

2. Construct $x = -\frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2}$ for different units and compare the results.

Definition: **homogenous equations** are equations all of whose terms are of the same dimension.

3. Prove that homogenous equations of the first and second dimensions have the same solution no matter what line is taken as unity.

4. Solve the following problems according to the Cartesian method.

a) Given the product of the extremes of three lines in continued proportion and their sum, find the lines.

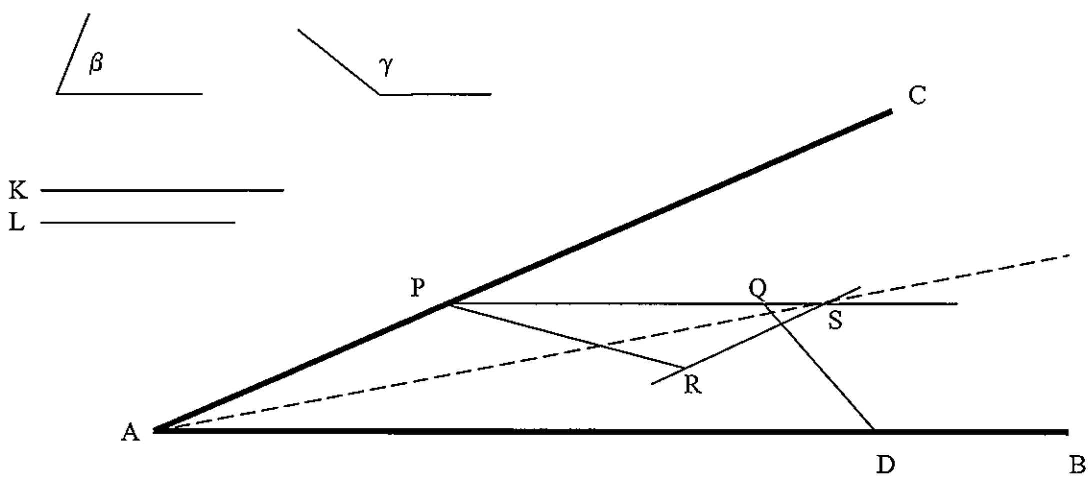
b) Given the product of the extremes and their differences, find the lines.

A Simple Locus Problem: (to accompany pp. 52-54).

Given: two lines in position, angles at which lines are to be drawn to the two lines, and the ratio which the two lines so drawn are to have. To determine the locus.

This is a two-line locus, formed by analogy with the three-, four-, etc.-line locus defined by Descartes and Pappus.

Let AB, AC be the two lines, meeting at A, $\angle\beta$ and $\angle\gamma$ be the given angles, and let K : L be the given ratio. Through any point P on AC, let PQ parallel to AB be drawn. From any point on PQ, let QD be constructed, making with AD an $\angle$QDA equal to $\angle\beta$. Let a fourth proportional M be taken to K, L, QD, such that K : L :: QD : M. Let $\angle$APR be constructed equal to $\angle\gamma$, and let PR equal M. Through R, let a parallel to AC be drawn, intersecting PQ at S. Let straight line AS be drawn. I say that AS is the required locus.



The proof is left for the student.

Locus-Problem Exercises: (to accompany pp. 52-54)

Find the line(s) which are the loci of points in a plane meeting the following qualifications:

a) At a given distance from a given point;

b) At a given distance from a given finite straight line;

c) At a given distance from a given circle;

d) Such that the lines from a point on the locus to the end points of a given straight line meet at right angles;

e) Such that the perpendicular from a point on the locus to a given straight line cuts that line into two segments whose product equals the square on the perpendicular;

f) Same as e) above except that the product has to the square on the perpendicular a given ratio;

g) Taking the problem from Pappus, VII, 155 (at the beginning of this manual), find the locus of points C such that they are the common end point of chords inscribed in circular segments on AB, which chords have to one another the given ratio E : F.

Schema of the Locus-Problem Solutions: (to accompany pp. 58-60)

The following schema may help to order the remarks which culminate on these pages:

Number of lines in a locus problem	**Number of dimensions in equation of curve solving the locus problem**	**Curve by means of which the equation is solved**	**Curve described by the equation, & hence, the solution to the locus problem**
3 or 4	2 in x and y	circles and straight lines	conic sections
5, not all //	2 in x or y	circles and straight lines	a special case (see pp. 55-56)
5-8	3-4	conic sections	the curve one degree higher than conics
9, not all //	4 in x or y	conic sections	a special case

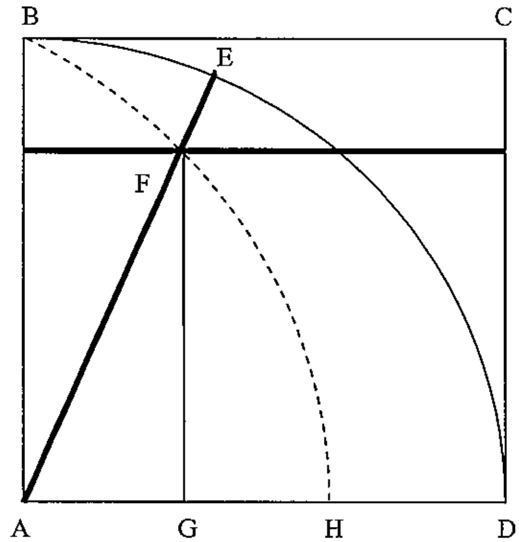
To make sense of this schema, we need to know what it means to solve a problem, and in particular an equation, by means of a curve, as well as what makes a set of curves higher ("more composed") or lower than another. To be sure, we have some idea as to the former, since we solved the problems leading to quadratic equations by means of circles and straight lines. We are presumably to imagine some more difficult problem, such as doubling the cube, which leads to an equation solvable only by the use of, say, a parabola instead of a circle, and a hyperbola in place of a straight line. In any case, Descartes is certainly correct when he states, in closing this book, that some general statements about the nature of curved lines have become necessary.

The Less Familiar Curves: (to accompany p. 62)

Spiral

Descartes has in mind the Arithmetic spiral, also known as the Spiral of Archimedes, who defines it as follows: If a straight line drawn in a plane revolve uniformly any number of times about a fixed extremity until it return to its original position, and if, at the same time as the line revolves, a point move uniformly along the straight line, beginning at the fixed extremity, the point will describe a spiral in the plane.

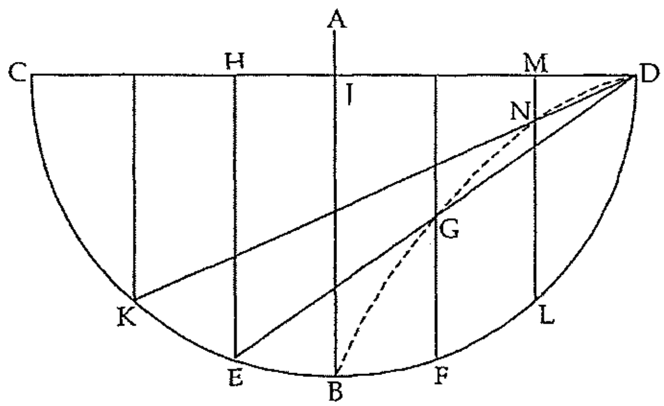
Quadratrix



Pappus, *Collection*, iv. 30, trans. Ivor Thomas:

> For the squaring of the circle a certain line was used by Dinostratus and Nicomedes and certain other more recent geometers, and it takes its name from its special property; for it is called by them the "quadratrix," and it is generated in this way.
>
> Let ABC be a square, and with center A let the arc BED be described, and let AB be so moved that the A remains fixed while B is carried along the arc BE; furthermore let BC, while always remaining parallel to AD, follow the point B in its motion along BA, and in equal times let AB, moving uniformly, pass through the ∠BAD (that is, the B pass along the arc BD), and let BC pass by the straight line BA (that is, let the B traverse the length of BA).
>
> Plainly then both AB and BC will coincide simultaneously with the straight line AD. While the motion is in progress the straight lines BC, BA will cut one another in their movement at a certain point which continually changes place with them, and by this point there is described in the space between the straight lines BA, AD and the arc BE a concave curve, such as BFH, which appears to be serviceable for the discovery of a square equal to the given circle. Its principal property is this. If any straight line, such as AFE, be drawn to the circumference, the ratio of the whole arc to DE will be the same as the ratio of the straight line of BA to FG; for this is clear from the manner in which the line was generated.

Cissoid

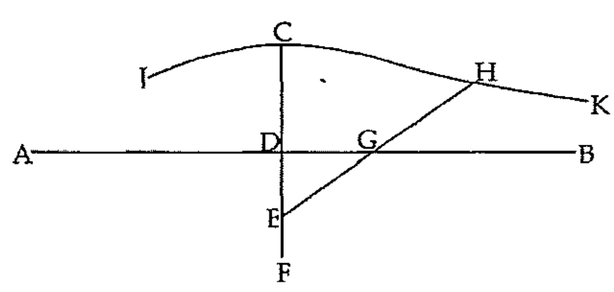


Pappus, *Collection*, iv. 32.

> If then on either side of B there be cut off equal arcs KB, BL, and LM be drawn through N parallel to AB, and DK be joined, ML, MD will again be mean proportionals between CM, MN.* If in this way more parallels are drawn continually between BD, and arcs equal to the arcs cut off between them and B are marked off from B in the direction of C, and straight lines are drawn from D to the points so obtained, such as DE, DK, the parallels between B and D will be cut in certain points, such as N, G in the accompanying figure. Joining these points with straight lines by applying a ruler we shall describe in the circle a certain curve, and if on this any point be taken at random, and through it a straight line be drawn parallel to BA, the line so drawn and the portion of the diameter cut off by it in the direction of D will be mean proportionals between the portion of the diameter cut off by it in the direction of the point C and the part of the parallel itself between the point C on the curve and the diameter CD.

*Note: That CM: ML:: ML: MD is obvious. Is ML : MD :: MD : MN? Hint: join CL and DL, and examine the angles in triangles MND & MDL.

Conchoid



Pappus, *Collection*, iv. 26.

> For the duplication of the cube a certain line is drawn by Nicomedes and generated in this way.
>
> Let there be a straight line AB, with CDF at right angles to it, and on CDF let there be taken a certain given E, and while the E remains in the same position let the straight line CDEF be drawn through the point E and moved

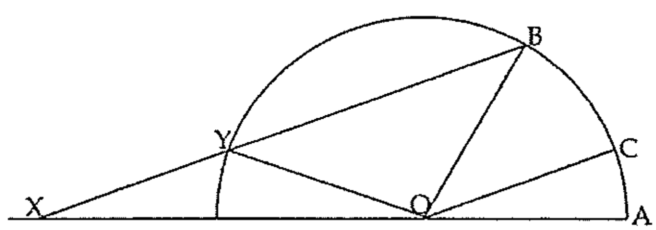
about the straight line ADB in such a way that D always moves along the straight line AB and does not fall beyond it while CDEF is drawn through E. The motion being after this fashion on either side, it is clear that the C will describe a curve such as JCK, and its property is of this nature: When any straight line drawn from E falls upon the curve, the portion cut off between the straight line AB and the curve JCK is equal to the straight line CD; for AB is stationary and E fixed, and when D goes to G, the straight line CD will coincide with GH and C will fall upon H. Therefore CD is equal to GH.

Similarly, if any other straight line drawn from the E falls upon the curve, the portion cut off by the curve and the straight line AB will make a straight line equal to CD. Now, says he, let the straight line AB be called the ruler, E the pole, CD the interval, since the straight lines falling upon the line JCK are equal to it, and let the curve JCK itself be called the first cochloidal line (since there are second and third and fourth cochloids which are useful for other theorems).

Nicomedes himself proved that the curve can be described mechanically, and that it continually approaches closer to the ruler—which is equivalent to saying that of all the perpendiculars drawn from points on the line JCK to the straight line AB, the greatest is the perpendicular CD, while the perpendicular drawn nearer to CD is always greater than the more remote; he also proved that any straight line in the space between the ruler and the cochloid will be cut, when produced, by the cochloid; and we used the aforesaid line in the commentary on the Analemma of Diodorus when we sought to trisect an angle.

The Conchoid (which Pappus here calls the "cochloid")[54] may be used to solve the problem of the trisection of the angle, one of the three famous problems said to be insoluble on the basis of Euclid's postulates. We will indicate how it might be done, proceeding analytically.

Let ∠AOB be the given angle, and let it have been trisected by CO, so that angle ∠COA = 1/3 ∠BOA. With O as center and with any radius OA, let a semicircle be drawn, and from B let a line BX be drawn parallel to OC, meeting OA produced at X, and cutting the circle in Y. Let YO be connected.



Then, taking parts such that ∠BOA is three parts, ∠BOC is two, and so is ∠YBO its equal. Likewise ∠BYO is two such parts, while ∠YXO, being equal to ∠COA, is one. But ∠BYO is equal to the sum of ∠YXO and ∠YOX. Hence the latter is also one part, and the triangle XYO is an isosceles triangle. Therefore XY equals YO, the radius of the circle.

If, then, we could find a line BX, of which the circle cuts off a segment XY equal to the radius of the circle, OA or OY, then, by conversion of the above analysis, the angle could be trisected. But this is the fundamental property of the conchoid; all lines from the pole to the ruler have the same distance cut off from the ruler by the curve.

The synthesis is left for the student.

[54] "Cochloid" and "conchoid" derive from κοχλοιδης (kochloidēs) and κογχοειδης (konchoidēs), meaning "snail-shaped" and "muscle/shell-shaped," respectively.

Problems Using the Cissoid and Conchoid:

Using the cissoid, find two mean proportionals between two given lines. For a greater challenge, use the conchoid to solve the problem.

Doubling the Cube Using the Hyper-Compass: (to accompany pp. 63-64)

Using the machine on p. 62 as a kind of hyper-compass, show how to double a given cube.

Remark: Let the side of the given cube be S, that of the cube that is to be made be X. Cubes are to one another in the triplicate ratio of their sides (*Elements* XI, 33). Thus,

cube S : cube X :: 1 : 2, by hypothesis,
And cube S : cube X :: S : 2S :: triplicate ratio of S : X .
If S: X :: X : Y :: Y : Z,
Then S : Z is the triplicate ratio of S : X (def. of trip. ratio)
Therefore Z = 2S.

Our problem, then, reduces to finding, X, the first of two mean proportionals between S and 2S. This will be the side of the square we are seeking. (Compare the beginning of book III of *The Geometry*, pp. 155-156 in the Dover edition.) If we can solve the problem, all that remains for us of the three unsolved problems is to square the circle!

The Machine: (to accompany pp. 64-65)

1. Prove that if the curve were referred to another point on AG besides A, the equation of the curve would remain of the same dimension.

2. Prove that if the reference line were inclined to AG at some angle, the equation of the curve would remain of the same dimension.

The Hyperbola and the Machine: (to accompany note 27, p. 65)

1. State a 4-line locus problem which gives this curve.

2. State a 3-line locus problem which gives the same curve.

The Length of LC^2: (to accompany p. 71)

1. Descartes distinguishes three simple cases for the length of LC^2.

a) $m^2 \pm ox + px^2/m$ where $o^2 = 4mp$ [Notice that the case of $-ox$ is an interpretation of the situation Descartes mentions where, he says, m^2 and px^2/m are both marked with the – symbol.]

b) px^2/m,

c) m^2

Construct the locus in each of the three cases.

2. Given $y = 2 - \frac{3}{4}x \pm \sqrt{\frac{9}{4}x^2 + 3x + 1}$, $\angle ABC = 90°$; construct the locus.

Equating Coefficients: (to accompany pp. 71-72)

The idea here is that, for any quantity x, if $Ax + B = Cx + D$, then $A = C$ and $B = D$. (The law here can be generalized to apply to larger exponents as well.) That Descartes may be thinking in terms of this law here (and elsewhere in *The Geometry*) can be gleaned from a remark he makes on pp. 80-81. The reader may find that it is not difficult to prove that we can equate coefficients.

Exercises Using Descartes's Method for Finding the Perpendicular (or "Normal") to a Curve: (to accompany pp. 77-79)

1. An easy example: Using Descartes method, construct PC for a circle.

2. Construct PC for a parabola.

On *The Geometry* as a whole:

In a letter written to Mersenne in 1637, Descartes says:

> I do not enjoy speaking in praise of myself, but since few people can understand my *Geometry*, and since you wish me to give you my opinion of it, I think it well to say that it is all I could hope for, and that in the *Dioptrics* and the *Meteorology* [the other two appendices of the *Discourse on Method*] I have tried only to persuade people that my method is better than the ordinary one. I have proved this in my *Geometry*, for in the beginning I have solved a question which, according to Pappus, could not be solved by any of the ancient geometers. Moreover, what I have given in the second book on the nature and properties of curved lines, and the method for examining them, is, it seems to me, as far beyond the treatment in the ordinary geometry as the rhetoric of Cicero is beyond the a, b, c of children"
>
> (*Oeuvres* [Paris, 1824], vol. 6, p. 294;
> translation from the Dover edition of *The Geometry*).