Tonality

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Introduction

IT IS often assumed today that the composition of good music is either a purely intuitive process, independent of the composer's training or knowledge, or, on the other hand, a process of following arbitrary rules and making arbitrary restrictions.

Paul Valéry has written, in an article on poetics:

> Not very long ago, all the arts were subject, each according to its nature, to certain obligatory forms or modes imposed on all works of the same genre; these could be and had to be learned, as we do the syntax of a language. . . . But gradually, and on the authority of very great men, the idea of a certain legality crept in and took the place of what had been, at first, recommendations of empirical origin. Reason put rigor into the rules. They were expressed in precise formulas; the critic armed himself with them; . . .[1]

These rules hardened into dogma instead of remaining flexible and subject to change, generalization, and improvement, the way a true science does. So of course they were broken, often for the mere sake of breaking them. The idea developed that one rule is as good as another: Wagner has Hans Sachs say in *Die Meistersinger,* "Set up your rules, and then follow them."

This book is an attempt to show that this is not true. It will be maintained that there are some rules which are better than others, relative to certain generally accepted aesthetic axioms. Specifically, it is the author's contention that the seven-tone diatonic scales, and combinations of these scales, are superior, relative to the axioms, to the twelve-tone chromatic scale, as tonal matrices from which to compose.

The reasoning will be as follows: "Root" will be defined quantitatively and operationally in such a manner that for any set of

[1] Valéry, Paul, "The Course in Poetics: First Lesson," *The Creative Process,* Mentor Books, N.Y., 1955, p. 92.

tones a procedure is given to determine whether it has a single root, many roots, or no root.

It will then be demonstrated that the diatonic major and the seven-tone harmonic minor sets have a single predominant root independent of their spatial or temporal distribution, and that the twelve-tone set (and the nineteen tone supra-scale of Yasser) do not.

The superiority of sets of tones having a single predominant root over those which do not will then be argued.

The natural phenomena which will be evoked as causes are (1) the existence of the overtone series, and (2) the existence of habit as a law of human nature.

That the diatonic set of tones has a single predominant root permits the development of a meaningful and consistent method of functional analysis of music which does not depend upon the analyser's intuition.

This fact also permits a meaningful definition of "tonality," as well as of a number of other words whose meaning has become confused and, in some cases, contradictory.

A theory of meter, related to the theory of pitch, will be presented.

I am more interested in truth than in originality, and may therefore say some things which have been said before, and which may be said again in the future. Heinrich Schenker and Paul Hindemith are the two music theorists to whom I am particularly indebted.

Chapter One

The Perceived Sound

IN ORDER that a tone may come into existence two things are required: a stimulus in the air, and a response of a person. It is the subjective response with which I am ultimately concerned. This response is a result of the whole past experience of the individual as well as of his physiology; it may vary from person to person and from time to time. Therefore, in order to learn something about it which is relatively independent of individual psychology, the attention must be directed to the common property of the events.

The most obvious common property is the stimulus: the set of sound waves which excite the response. These can be registered by an impartial machine, and can be described without ambiguity. A curious thing happens when this experiment is made, that is, when a machine registers the sound waves emitted in musical performance by instruments without fixed pitch.

> . . . the most significant thing about the result of this experiment is that it required the intervention of the measuring instrument to reveal these grotesque distortions of pitch, these false tones. The audience, which included experienced musicians, had not noticed them at all.[1]

Thus a large error, in this case as much as 80 cents (a semi-tone is 100 cents) relative to an ideal tuning, passed by unnoticed by a musical audience. For music, like history, is a process in which each additional tone or event may alter to some extent the significance, syntactical or symbolic, of the tones which preceded it. The context in which the pitches appear affects the heard pitch relationships themselves, and hence the integers which express them. However, unless we first consider sets of tones out of context there is no hope of ever being able to attack the more difficult problem of sets of tones in context.

[1] Zuckerkandl, Victor, *Sound and Symbol,* Bollingen Foundation, N.Y., 1956, p. 79.

I speak of sets of tones here rather than of chords, keys, or scales, for the word "chord" implies simultaneity, the word "scale" implies temporal ordering, and the word "key" has acquired many contradictory meanings. By "set" I merely mean a group of tones whose frequencies are related to each other as are the integers which represent them, quite apart from the temporal order of occurrence of the tones, whether it be simultaneous or successive.

Many sets of tones can be expressed as sets of relatively prime integers whose greatest common divisor is 1. For example, the major triad is expressed by the numbers 4:5:6, where 4 corresponds to the heard frequency of the lowest tone, 5 to that of the middle tone, and 6 to the frequency of the highest tone. However, there are some sets which cannot be unequivocally translated into sets of integers. For if the sets are found high in the overtone series [2] there will be several different sets of integers which may represent them, and it is not always possible to decide unequivocally which is the best representation. Perception is not simply the registering of sense impressions; a certain amount of interpretation enters in as well. The clue from the outer world, sense impression, is interpreted by most individuals as being something familiar whenever this is possible. The musical intervals with which everyone is unavoidably (albeit unconsciously) familiar, are those intervals found between the lower partials of the overtone series.

Figure 1 is a representation of the first sixteen partials of the overtone series. To each integer there is associated a tone, but due to our tuning systems only those tones which have no other prime fac-

[2] Whenever a musical tone is sounded, whether produced by a vibrating string or an air column, what actually occurs is a whole set of vibrations which the ear synthesizes into one sound. For the string vibrates simultaneously, or nearly simultaneously, as a whole, in halves, in thirds, etc. ad infinitum. Each of these vibrations produces a partial tone. If the frequency produced by the string as a whole is 1, then the frequency produced by half the string is 2, and by a third of the string is 3, and so on. This physical manifestation of the natural numbers was discovered by the physicist Joseph Sauveur in the seventeenth century.

"I was made to observe that especially at night one may hear from long strings not only the principal sound but also other small sounds . . . such that the number of vibrations is a multiple of the number for the fundamental sounds . . . I concluded that the string in addition to the undulations it makes in its entire length so as to form the fundamental sound may divide itself in two, in three, in four, etc. undulations . . ."

Joseph Sauveur, "Systême general des intervalles des sons, & son application à tous les systemes & a tous les instrumens de musique," Mém. acad. sci. Paris 1701.

tors than 1, 2, 3, and 5 can be represented accurately by musical notation. All other tones are inaccurate, and are in parenthesis.

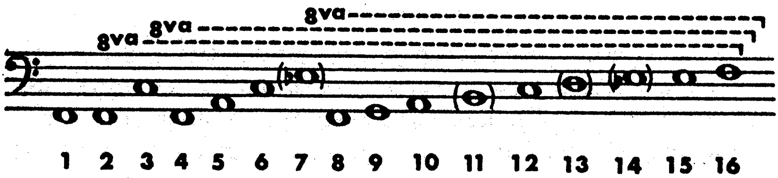


Figure 1. The First 16 Partials of the Overtone Series, in Musical Notation

The overtone series is relevant to music, because musical tones, including those of the human voice, are produced by the impingement of a set of pulses in the air, upon the hearing organ of a person.[3] A pulse is a periodic alternation between high and low air pressure. The wave lengths of the set of pulses or partial tones which constitute a single musical tone are related to each other as are the positive integers, 1, 2, 3, . . . etc. Each musical tone consists of a theoretically infinite quantity of these wave trains. Of course the set of pulses which our instruments can measure is always a finite set. How many partials are perceived depends upon the sensitivity of the instrument.

The partial tones whose frequencies, or quantity of pulses per second, correspond to the smaller numbers, are louder, or have a greater amplitude, by and large, than do those pulses whose frequencies correspond to the higher numbers. (In Culver's *Musical Acoustics* graphs are displayed which demonstrate the loudness of the various partials in different instruments.[4]) Furthermore, the smaller the number of pulses per second of a partial tone, the more that partial is reinforced, or confirmed, by partials at upper octaves. (See Figure 1). The octave of a partial tone is represented by a number which is twice the number of the partial. From a purely numerical standpoint, the small numbers are more important than the larger numbers, because the larger numbers are formed from sets of the smaller numbers, and thus depend upon the smaller numbers for their very existence. Hence the lower

[3] Euler, Leonhard, *Letters to a German Princess,* Vol. 1, Harper, New York, 1833, p. 39.

[4] Culver, Charles A., *Musical Acoustics,* McGraw-Hill, New York, 1956, p. 162.

partials of a tone are more important to us than are the upper partials.

What we hear as a difference in tone quality is caused by the difference in the distribution and amplitudes of the partials of the tones. We are not immediately aware of the nature of the stimulus; it took the physicists to discover that. Similarly, we are not immediately aware of the atomic structure of a table, nor of the periodic nature of light rays. This does not, however, prevent us from differentiating between different colors of light, or perceiving the wood of the table as being different from the metal of the typewriter upon it.

Since the seventeenth century physicists and music theorists have been aware of the existence of the overtone series. But even before that the music theorists, as early as Pythagoras and as late as Descartes, dealt with tones which could be produced by successive divisions of a string into equal parts. The senario, or the numbers from 1 to 6 with which Zarlino, Descartes and other theorists were concerned, corresponds to the first six partials of the overtone series. The representation of the intervals of music by the ratios of two numbers can be found all throughout the history of music theory. In the nineteenth and early twentieth centuries, however, many theorists denied the relevancy of the overtone series to music, thinking that by concentrating on the response to the periodic stimuli they could disregard the stimuli themselves. This is part and parcel of the general tendency of the time to emphasize the subjective aspects of experience over the objective. But responses are, except in the case of hallucinations, dependent on their stimuli, and therefore, if the response is relevant to music, so also is the stimulus. Today some music theorists are again asserting the relevancy of the overtone series to music.[5,6,7]

Thus the assumption concerning human psychology will be made that within a certain as yet undetermined range the individual perceives the periodicities which stimulate him as the small ratios with which he is most familiar, due to his unavoidable condi-

[5] Hindemith, Paul, *The Craft of Musical Composition,* Associated Music Publishers, N.Y., Vol. I, 1942, p. 15.

[6] Redfield, John, *Music, A Science and an Art,* Alfred Knopf, N.Y., 1930, p. 46.

[7] Bobbitt, Richard, *The Physical Basis of Intervallic Quality and its Application to the Problem of Dissonance,* Journal of Music Theory, Vol. III, 1959, Yale School of Music.

tioning to the overtone series. This statement can be justified without recourse to any more dubious law of human nature than "people are creatures of habit." No further attempt at justification of this assumption will be made: however, it is implicitly assumed by anyone who assigns integral ratios to the intervals of music, and this includes nearly all the music theorists of the past and present, for the actual pitches of equal temperament are incommensurable.

I shall assume that people tend to perceive equal-tempered intervals as pure intervals. By "pure" intervals is meant those intervals whose frequencies are related to each other as are the tones of the lower partials of the overtone series. The farther up the overtone series the interval is to be found, the more difficult it is for the listener to identify the interval, for the higher partials are softer and less familiar to us than are the lower partials. Furthermore, until the twentieth century, most of our scales have been constructed by combining the intervals between the lower partials, rather than those between the upper partials; so our past experience with scales also confirms this tendency.

There is no uncertainty or doubt as to the correlation of the integers 1:2 with the octave. No one questions the fact that this is the paradigm ratio which the listener perceives, although the actual stimulus may deviate slightly from it. But with other intervals it is not always possible unequivocally to determine the paradigm ratio which the ear perceives. Consider the minor seventh. The ratio between the fourth and seventh partials, 4:7, does not exist in any of the tunings used in Western music. The just scale contains two different minor sevenths, 5:9, and 9:16, while the minor seventh of the Pythagorean scale is 9:16. Which is the paradigm to which the ear corrects the irrational interval of the equal-tempered scale, 4:7, 5:9, or 9:16? Arguments can be made in favor of each. An intellectual problem as well as a sensory one is involved in arriving at the integers which are the paradigms of the intervals we hear. Figure 3, page 17, shows, among other things, the different numerical ratios which can be assigned to various familiar sets of tones.

It is possible to arrive intellectually at numerical relationships which will represent any set of tones by the Pythagorean system of tone derivation, that is, by combination of 1:2 octaves and 2:3 fifths. However, the integers arrived at by this process may be very large. Nevertheless, no intellectual doubt or compromise is necessary in order to do this. The compromise does not appear until one of two things is desired: the 4:5 major third, or enharmonic equivalence.

If the 4:5 major third is desired instead of the 64:81 Pythagorean third, the just scale results. It incorporates more of the overtone series by including the fifth partial as well as the second, third, and fourth partials, thus allowing the development of triadic harmony, but in doing so it eliminates the unequivocality of the Pythagorean system. For in the just diatonic scale more than one ratio is associated with all intervals except the octave, the major third, the minor sixth, major seventh, and minor second.

If enharmonic equivalence is asserted, then two pitches which are not the same, e.g., E-sharp and F, are said to be the same, and the quantity of pitches which fill in the octave is restricted to the finite number 12. This operation transforms the unequivocal Pythagorean scale into the equal-tempered scale, all of whose intervals are irrational, and therefore do not serve as paradigms. There is much music where any problem of enharmonic equivalence can be eliminated by transposition of the whole piece into another key. However, if the whole circle of twelve keys is involved in the composition, then enharmonic equivalence is an unavoidable phenomenon; it is something real, not just notational.

Once either of these two things is desired there must be an intellectual compromise, for no power greater than 0 of one prime number ever equals a power of another prime number; thus no tone derived by successive 1:2 octaves will ever be identical with any tone derived by successive 4:5 thirds. The equal-tempered scale provides a wonderful compromise, which allows us to have our cake and eat it too; by asserting the lie that twelve fifths equal seven octaves it achieves at one and the same time enharmonic equivalence and a close approximation to the pure third, at a slight sacrifice of the purity of the fifth. Neither the deviation from the pure fifth or the pure third is enough to prevent the ear from being able to perceive the irrational interval as the pure one.

While the equal-tempered scale has been of great benefit to composers, it has not always been of such benefit to theorists. For there has been a tendency to confuse the means with the end, to substitute the equal-tempered set itself, and decimal approximations to its intervals, for the groups of diatonic rational scales which it approximates, thus losing sight of those rational relationships which justify its very existence. As far as perception is concerned, the equal-tempered scale is a wonderful compromise, but as far as conception is concerned, it clouds the important relationships. However, unless percept and concept are confused into one, equal temperament need cause no difficulties for composer or theorist.

The equal-tempered set enables the composer to grasp as much as possible of the overtone series without losing contact with the fundamental. It does not sacrifice the lower intervals of the overtone series to the higher ones; it even gives a decent approximation to the next prime partial above the fifth one, the seventh partial. Furthermore, it is likely that *all* the other intervals formed between partials of the overtone series are represented in the equal-tempered scale within a range of accuracy smaller than the error in the Zuckerkandl experiment, referred to previously. Figure 2, below,

Name	Ratio	Decimal Approximation	Closest Equal-tempered Equivalent	Decimal Approximation	Discrepancy Ratio
P. 8th	2:1	2.000	2	2.000	1.000
P. 5th	3:2	1.500	$2^{7/12}$	1.498	1.001
M. 3rd	5:4	1.250	$2^{1/3}$	1.260	1.008
m. 7th	7:4	1.750	$2^{5/6}$	1.782	1.019
M. 2nd	9:8	1.125	$2^{1/6}$	1.122	1.010
(A. 4th)	11:8	1.375	$2^{1/2}$	1.414	1.029
(m. 6th)	13:8	1.625	$2^{2/3}$	1.587	1.023
M. 7th	15:8	1.878	$2^{11/12}$	1.888	1.005
(m. 2nd)	17:16	1.062	$2^{1/12}$	1.059	1.003
(m. 3rd)	19:16	1.188	$2^{1/4}$	1.189	1.000
(P. 4th)	21:16	1.312	$2^{5/12}$	1.335	1.016
(A. 4th)	23:16	1.439	$2^{1/12}$	1.414	1.017
(m. 6th)	25:16	1.562	$2^{2/3}$	1.587	1.015
M. 6th	27:16	1.688	$2^{3/4}$	1.682	1.002
(m. 7th)	29:16	1.812	$2^{5/6}$	1.782	1.017
P. 8th	31:16	1.939	$2^{11/12}$	1.888	1.028

An equal-tempered semitone = 1.059

Figure 2. A Comparison Between Pure Intervals and Equal-Tempered Intervals

shows the relationships between (1) the intervals found between the fundamental, or octave transposition of the fundamental, and the first 31 partials of the overtone series, and (2) the equal-tempered equivalent intervals. The last column gives a measure of the discrepancy between the equal-tempered interval and the pure interval. The biggest discrepancy occurs between the eleventh partial and its equal-tempered representation. It is close to a quarter-tone, while the error in the Zuckerkandl experiment referred to earlier was nearly a semitone.

The equal-tempered twelve-tone scale introduces the fruitful compromise which permits unlimited modulation and hence the development of Western music, but it does so at the price not only of slight inaccuracy of pitch, but also of intellectual equivocality of intervals. For in order to arrive at the rational numerical representation of an equal-tempered interval we must consider the context, which may be infinitely varied. Thus the potential wealth of the equal-tempered scale is made possible only by the sacrifice of univocal definition of intervals.

Figure 3, page 17, is a table of familiar sets of tones, with some of the different integral numerical ratios which can be associated with them. The first column contains integers derived from the Pythagorean 12-tone scale; they represent tones derived solely by the superposition, or compounding, of 2:3 fifths and 1:2 octaves. The second column contains integers found in the just diatonic scale. The third column contains the smallest possible integers arrived at by combining 2:3 fifths, 4:5 major thirds, and 5:6 minor thirds. The fourth column contains integers derived directly from the overtone series, beginning with the lowest possible partials which could represent the particular set in question, and ending with the sets arrived at by the other columns. If this terminus were not placed upon the numbers of Column IV, there would be an infinite quantity of candidates for each set among the upper partials, since if we go high enough we can find any set of integers.

It can be seen from Figure 3 that the intervals found between the lower partials have fewer different ratios associated with them than do the sets found between the higher partials, each of which has several different sets of integers representing it. The fact that we have a single word for these different mathematical relationships is itself a reflection of human perception thresholds, for we give a single word to things which appear to be the same to us.

Name	Pythagorean Scale	Just Diatonic Scale	Combination	Overtone Series
P. 8th	1:2	1:2	1:2	1:2
P. 5th	2:3	2:3 27:40	2:3	2:3
P. 4th	3:4	3:4 20:27	3:4	3:4
M. 6th	16:27	3:5 16:27	3:5	3:5
m. 3rd	27:32	5:6 27:32	5:6	5:6
M. 3rd	64:81	4:5	4:5	4:5
m. 6th	81:128	5:8	5:8	5:8
m. 7th	9:16	5:9 9:16	5:9	4:7 5:9
M. 2nd	8:9	8:9 9:10	8:9	7:8 8:9
M. 7th	128:243	8:15	8:15	8:15
m. 2nd	243:256	15:16	15:16	15:16
A. 4th	512:729	32:45	25:36	5:7 7:10 8:11 9:13 11:14 etc.
D. 5th	729:1024	45:64	25:36	5:7 7:10 8:11 9:13 11:14 etc.
M. triad	64:81:96	4:5:6	4:5:6	4:5:6
m. triad	54:64:81	10:12: 15	10:12:15	6:7:9 10:12:15
D. triad	729:864: 1024	45:54: 64	25:30:36	5:6:7 11:13:15 12:14:17 etc.

Figure 3. Different Integral Representations of Familiar Sets of Tones

Name	Pythagorean Scale	Just Diatonic Scale	Combination	Overtone Series
A. triad	4096:5184: 6534	none	16:20:25	7:9:11 8:10:13 9:11:14 etc.
M.m. 7th	576:729: 864:1024	36:45: 54:64	20:25: 30:36	4:5:6:7 12:15:18:21 20:25:30:36
m.m. 7th	54:64: 81:96	10:12: 15:18 27:32: 40:48	10:12: 15:18	10:12:15:18
½D. 7th	729:864: 1024:1296	45:54: 64:80	25:30: 36:45	5:6:7:9 11:13:16:20 etc.
D. 7th	too big	none	75:90: 108:125	10:12:14:17 etc.
M. 9th	576:729: 864:1024: 1295	36:45: 54:64: 80	36:45: 54:64: 80	4:5:6:7:9 12:15:18:21:27 20:25:30:36:45 36:45:54:64:80
m. 9th	too big	none	60:75: 90:108: 128	8:10:12:14:17 20:25:30:36:42 36:45:54:64:75 60:75:90:108:128

Figure 3. Different Integral Representations of Familiar Sets of Tones (continued)

Although at first it appears to be impossibly complicated to discuss properties of different sets when confronted by all their different possible numerical representations, there are, nevertheless, only a finite quantity of relevant possibilities, and in the event that all the numerical representations of a set have the same properties, we can come to unequivocal conclusions about the nature of the set.

The table shows that seconds, sevenths, the augmented fourth, and the diminished fifth differ from the other intervals, for column IV shows several possibilities for each of these, and only one for the others. These intervals have an inherent ambiguity not possessed by the others, for a number of possible ratios exist for them, each of which is lower in the overtone series than the numerical representations occurring in the just or Pythagorean scales. Which of these ratios the ear will perceive depends upon the musical context. For example, the major second between the two lowest tones of a dominant seventh chord in its third inversion is likely to be heard as 7:8, because the dominant seventh chord is a gestalt which is in itself a representation of the overtone series as a whole; therefore the seventh of the chord will be likely to be heard as the seventh partial of the overtone series. On the other hand, a major second occurring in a diatonic melody is more likely to be perceived as 8:9, its value in the diatonic scale. This problem does not arise concerning thirds, sixths, perfect fifths, and perfect fourths, for in their cases the lowest overtone series representation is reinforced, rather than contradicted, by the just scale, since the just scale is constructed by combining these very intervals. (There exist two varieties of perfect fifth, perfect fourth, major sixth, and minor third in the just scale, but they are related to each other as norm and variant due in part to the relative frequency of their distribution in the scale.) In the case of the octave, perfect fourth and perfect fifth, the overtone series interval is reinforced still further by the Pythagorean scale, since that scale is constructed by combining of these intervals.

Thus it can be seen that any evaluation of the properties of sets of tones must take into consideration not only the numerical ratios of the sets, but also the uniqueness of the association of the ratio to the interval itself.

Summary

1. Both stimulus and response are necessary for a tone to come into existence. Since the response is a function of the stimulus as well as many other possibly irrelevant factors, I choose to concentrate upon the stimulus.

2. While it is true that the musical context effects the nature of the tone as well as its relations to other tones, it is also true that unless we artificially isolate tones and relationships between them we can have no hope of arriving at any understanding of the gestalt.

3. A set of tones is a group of pitches (which may be represented by integers) apart from the order of their occurrence in time.

4. Because we are creatures of habit and are subject to conditioning, we are unavoidably conditioned to the overtone series due to its physical presence in musical sounds; for this reason (and possibly for others as well) we unconsciously tend to correct the stimuli to the intervals between the lower partials.

5. The just scale sacrifices the intellectual unequivocality of the Pythagorean scale to the more accurate representation of the lower partials.

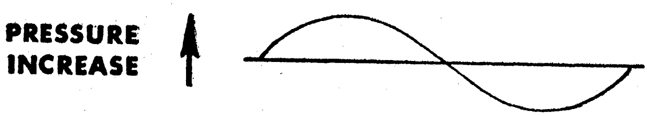
6. Equal temperament gives an adequate approximation to all intervals.

7. Any evaluation of sets of tones must take into account the equivocality of its integral representation.

Chapter Two

Consonance

IN ORDER to be able to represent sound waves as points in time we must first establish a pressure norm, represented by the horizontal line in the following picture:



A single wave length can be represented by its end points: . . , and a frequency can be represented spatially by a series of equally spaced points.

Figure 4 (p. 22) shows the pattern of the first twelve partial tones of the overtone series as they occur in the unit time interval, each row of dots standing for a partial tone. The figure shows only the first twelve partials, since that is all the page will hold; it could be extended indefinitely, for the overtone series is a temporal manifestation of the infinite set of the natural numbers.

In the physical world no two different occurrences of sets of tones have the identical consonance value, for the perceived complexity of the wave pattern varies with range, instrumentation, intensity, number of partials, number of combination tones, and the state of the perceiver, as well as with other more or less irrelevant factors which affect the perception of similar stimuli at different times. But I am here considering the ideal conditions of which the actual occurrence is a variant. This ideal condition is the unity, or common property, or essence, of the diversity of experience. The theorist must emphasize these common properties, for if he chooses to emphasize the differences, no science, even no language, is possible.

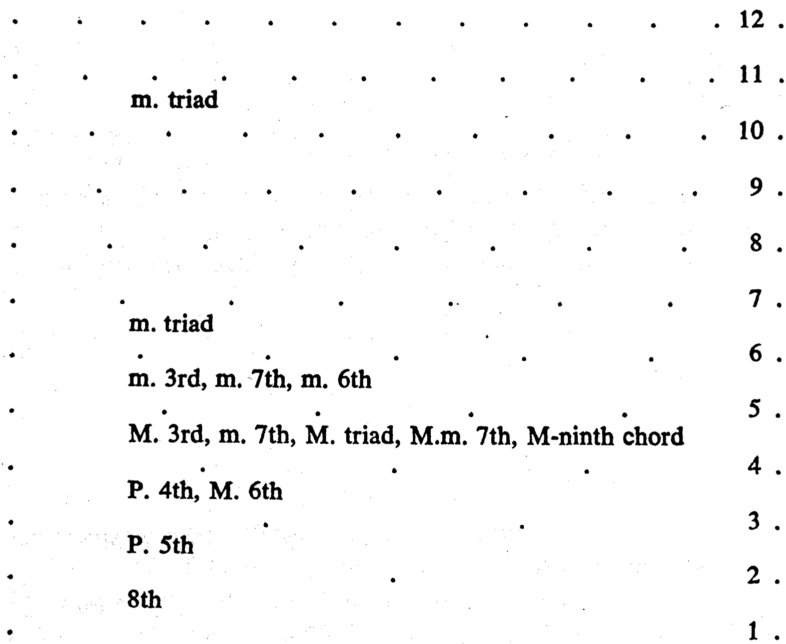


Figure 4. The First Twelve Partials of the Overtone Series as Points in Time, and the Position of some Sets of Tones in it.

"Consonance" is a name for such a common property of experience. Psychologically, "consonance" refers to the property of some sets of tones of sounding as though they belonged together. An attempt will be made here to choose objective phenomena which provide the measure of consonance in this sense.

Any set of integers determines a single unit integer, 1, of which the others are compounded. If these integers are interpreted as frequencies of partial tones of an overtone series, then this 1 is the frequency of the tone which is the fundamental of that overtone series. Therefore, any set of partials, when represented by integers, determines a single unique overtone series, the frequency of whose fundamental is 1. Now, we represent not only sets of partials, but sets of compound tones as well, by integers which, in the case of the compound tones, represent the frequency of the fundamental of each tone. (All tones except those produced by instruments specifically designed for the purpose, are compound tones.) The following is a representation of the 4:5:6 major triad:

. . . . C

. . . . A

. . . . . . F

These three tones determine another tone, an F two octaves below the F of the chord, which has only one vibration in the same amount of time that the F of the triad has four vibrations, the A five vibrations, and the C six vibrations. This low F is related to the tones of the triad as the fundamental of an overtone series of which they are the fourth, fifth, and sixth partials. We shall call this tone the fundamental of the set.

When the integers which represent the tones of the set are relatively prime, then 1 is their greatest common measure. The greatest common measure of the integers thus represents (1) the smallest quantity of waves which take place in the unit time interval under consideration, (2) the largest wave length, and (3) the frequency of the fundamental of the set. The least common multiple [1] of the numbers represents, conversely, the smallest interval of time determined by the set and the largest quantity of time intervals determined by the set. The ratio of the least common multiple to the greatest common divisor equals the number of times which the smallest time interval determined by the set must be repeated in order to equal the largest time interval.

When any set of wave lengths are combined they create a long wave pattern, which will be repeated identically as long as the tones are sounding. The frequency of the long wave pattern is the same as that of the fundamental of the set. I will define the consonance of any set of tones to be a function of two different properties of the set: (1) the position of the set in the overtone series of the fundamental it determines, and (2) the inner complexity of the long wave pattern itself. The consonance increases as the distance of the set from its fundamental decreases. The consonance decreases as the inner complexity of the wave pattern increases.

By this definition, consonance is a matter of degree, not kind. I shall say of two different sets of tones that one is more consonant than the other; not that one is absolutely consonant and the other

[1] The least common multiple of a set of numbers is the smallest number which is divisible by all the given numbers.

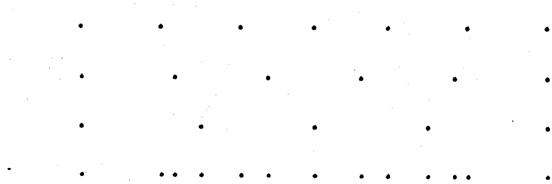
is absolutely dissonant. This substitutes a spectrum for a black and white distinction. Dissonance increases as consonance decreases, and vice versa. A chord can be dissonant or consonant in the same sense that a thing can be big or little.

The upper partials of each tone of the set enter into the total evaluation of consonance, as Helmholtz has pointed out,[2] but since they themselves are functions of their fundamentals, only the relationships between the tones as fundamentals will be considered here in the establishment of the basic theory. The upper partials of the individual tones influence the complexity of the pattern (Property II) but not the position of the pattern in the overtone series of the fundamental it determines (Property I).

The measure of Property I, the distance of the lowest tone of the set from the fundamental of the set, is given by the smallest number in the integral representation in lowest terms of the set. The measure of Property II, the inner complexity of the pattern, is given by the ratio of the least common multiple of the integers to the greatest common divisor of the integers.

For example, consider the 4:5:6 major triad. Its position in the overtone series it determines is given by 4, the number of vibrations the lowest tone makes in the same time that the fundamental determined by the tones makes one vibration. The measure of the complexity of the pattern is the least common multiple divided by the greatest common divisor. Since the numbers are relatively prime, the greatest common divisor is 1. The least common multiple is 60 and the ratio of the two is 60/1 or 60.

The following picture shows the major triad and, underneath it, the points all superimposed on the same line:



(Note that if the first, second, and third partials had been included in this picture, the superimposed points would be no different.) The largest line segment which will divide all the segments is a

[2] Helmholtz, Hermann, *Sensations of Tone,* Longmans, Green, London, 1885, p. 182.

segment equal to 1/60th of the whole line. 60 of these segments constitute the whole line. Thus 60/1, or the least common multiple divided by the greatest common divisor, is a measure of the complexity of the pattern.

Property I, that is, the distance from the fundamental determined by the set to the lowest tone of the set, is important due to the fact that our experience with the lower partials is far greater than our experience with the upper partials; therefore, the lower the position of the set relative to its fundamental, the more readily we are able to grasp its existence as a set. But this is not enough, for if it were, then the major third, major triad, dominant seventh chord, and the dominant major ninth chord would all be equally consonant. Therefore the inner complexity of the set itself must be considered, for indeed, if the complexity were too great the set would not be perceived as an entity in itself; the pattern would not be recognized as a pattern at all, only as chaotic non-periodic confusion, thus rendering imperceptible the fact that something is being repeated.

Figure 4, page 22, shows the positions of various familiar sets relative to the fundamental which generates them. The name of the set is written at the level of the lowest pitch of the set. In this figure the fundamental is held constant. Figure 5, page 26, shows a comparison of some sets by holding the lowest tone of the set, rather than the fundamental, constant.

The importance of Property I is shown by the fact that in some cases, namely, when the lowest tone is not a multiple of 2, Property I is independent of octave separation of the tones of the set. This also explains the apparent relative consonance of intervals normally considered dissonant, when they are separated by several octaves, without any necessity for resorting to the clashing of the upper partials. For if the lowest tone of an interval is a power of two, then when the upper tone is transposed up an octave, the ratio is simplified. For example, consider the ratio 4:5, the major third. If the upper tone is transposed up an octave the ratio become 4:10, or 2:5, which is simpler and more consonant than the original ratio.

Different sets with different letter names may have the same value for Property I and a different value for Property II. For example, the 4:5 major third and the 4:5:6 major triad both have the value 4 for Property I, while their values for Property II are 20 and 60 respectively.

Name	Ratio	Fund.	Picture
			. . . F
P. 8th	1:2	F	. . F
			 C
P. 12th	1:3	F	. . F
			...... A
M. 10th	1:5	F	. . F
			..... F
two 8ths	1:4	F	. . F
			 C
P. 5th	2:3	F	. . . F
			. . . . . . B♭
P. 4th	3:4	B♭	. . . . . F
			. . . . A
M. 3rd	4:5	F	. . . . . . F
			. . . . G
M. 2nd	8:9	F	. . . . F
			 C
			. . . . A
M. triad	4:5:6	F	. . . . . . F
			 C
			. . . . A♭
m. triad	10:12:15	D♭	. . . . F
			. . . . C♭
			. . . . A♭
D. triad	5:6:7	D♭	. . . . F
			. . . . E♭
			 C
			. . . . A
M.m. 7th	4:5:6:7	F	. . . . . . F

Figure 5. Comparison of Some Familiar Sets of Tones

Different sets with different letter names may have the same value for Property II and a different value for Property I. For example, the 4:5:6 major triad and the 10:12:15 minor triad both have 60 as a measure of their complexity. Also, the 25:30:36 diminished triad and the 20:25:30:36 dominant seventh chord have the same measure of complexity, or Property II, as do the diminished seventh and dominant minor-ninth chords. This is one reason why the diminished chords act as incomplete dominants and are often considered as such.

Figure 6, page 28, gives the values of Property I and Property II of some familiar sets of tones by way of illustration. A coefficient of total consonance is also given in this table, which is derived by taking the sum of Property II and 100 times the value of Property I. It must be remembered that consonance diminishes as the numbers increase.

Summary

1. Tones can be represented by equally spaced points on a line, thus giving a spatial representation of a temporal phenomenon.

2. The overtone series is a set of such sets of points, but since it is infinite the picture must always be incomplete.

3. To every set of tones which are represented by integers, it is possible to assign a consonance value which depends upon the position of the set relative to its fundamental, and on the ratio between the least common multiple and the greatest common divisor of the set. Differences in consonance are quantitative, not qualitative.

Name	Numbers	Letters	Consonance Properties I	II	Total Consonance Coefficient (100xI) +II
Octave	1:2	FF	1	2	102
P. 5th	2:3	FC	2	6	206
P. 4th	3:4	CF	3	12	312
M. 6th	3:5	FD	3	15	315
M. 3rd	4:5	FA	4	20	420
m. 3rd	5:6	DF	5	30	530
m. 6th	5:8	AF	5	40	540
m. 7th	5:9	FE$^{\flat}$	5	45	545
M. 2nd	8:9	E$^{\flat}$F	8	72	872
M. 7th	8:15	FE	8	120	920
M. 2nd	9:10	E$^{\flat}$F	9	90	990
m. 7th	9:16	FE$^{\flat}$	9	144	1044
m. 2nd	15:16	EF	15	240	1740
A. 4th	18:25	FB	18	450	2250
D. 5th	25:36	BF	25	900	3400
A. 4th	32:45	FB	32	144	3344
D. 5th	45:64	BF	45	288	4788
M. triad	4:5:6	FAC	4	60	460
	5:6:8	ACF	5	120	620
	3:4:5	CFA	3	60	360

Figure 6. Consonance Values

Name	Numbers	Letters	Consonance Properties I	Consonance Properties II	Total Consonance Coefficient (100xI) +II
m. triad	6:7:9	FA♭C	6	126	726
m. triad	10:12:15	FA♭C	10	60	1060
	12:15:20	A♭CF	12	60	1260
	15:20:24	CFA♭	15	120	1620
M.m. 7th	4:5:6:7	FACE♭	4	420	820
M.m. 7th	20:25:30:36	FACE♭	20	900	2900
	25:30:36:40	ACE♭F	25	1800	4300
	15:18:20:25	CE♭FA	15	900	2400
	18:20:25:30	E♭FAC	18	900	2700
D. triad	5:6:7	FA♭C♭	5	210	710
D. triad	25:30:36	FA♭C♭	25	900	3400
A. triad	16:20:25	FAC♯	16	400	2000
m.m. 7th	10:12:15:18	FA♭CE♭	10	180	1180
½ D. 7th	25:30:36:45	FA♭C♭E♭	25	900	3400
M. 9th	20:25:30:36:45	FACE♭G	20	900	2900
D. 7th	125:150:180:216	FA♭C♭E♭♭	125	5400	17900
m. 9th	100:125:150:180:216	FACE♭G♭	100	5400	15400
cluster	8:9:10	FGA	8	360	1160

Figure 6. Consonance Values (continued)

Chapter Three

Roots

NEARLY EVERYONE from Pythagoras to Bobbitt [1] has accepted the fact that consonance is in some way dependent upon the integers which express the relative frequencies of a set of tones. But what of the idea of root? Does it also have a basis in the physical world of sound, as consonance does, or is it just a terminological convention which once was observed and was later discarded in favor of other equally arbitrary terminological conventions? Is the root the lowest tone of a chord composed of superposed thirds when it is arranged in closed position? Or is it, as Hindemith maintians,[2] a property of intervals as well, and therefore equally present in nearly all sets of tones?

My answers to these questions are that roots are not arbitrary terminological conveniences, that they are present in the physical world as properties of some intervals and some larger sets of tones, and that they are perceived in music as being relatively important tones, or tones which are more important than their neighbors. Some sets of tones have roots and others do not. Some sets of tones have several conflicting roots which cancel out each other. Some sets of tones have stronger roots than other sets. The root of a set of superposed thirds is not necessarily the lowest tone; such a set may or may not have a root.

As we have seen, all sets of tones can be expressed as sets of integers. Thus there exists a tone represented by 1 which generates all the tones of the set; it is analogous to the fundamental of an overtone series. A set will be said to have a root if this fundamental tone, or an octave transposition of it, is contained in the

[1] Bobbitt, Richard, *The Physical Basis of Intervallic Quality and its Application to the Problem of Dissonance,* Journal of Music Theory, Vol. III, 1959, Yale School of Music.

[2] Hindemith, Paul, *The Craft of Musical Composition,* Schoot and Co., London, 1942, Vol. I, p. 87.

set. Every octave transposition of the fundamental is a power of 2, the fundamental itself being 2°, or 1. This tone, if it exists, is the root of the set.

Consider the simplest case, where the sets have only two members. The root of such a set, if it exists, will be, then, the tone which is represented by the smallest power of 2. If the interval contains no power of 2, then it does not contain its fundamental or an octave of its fundamental, and therefore it has no root. For example, the 2:3 fifth has its lowest tone as root, for 2 equals 2^1 and 3 is not a power of 2. The minor third, 5:6, has no root, for neither 6 nor 5 is a power of 2.

Figure 7 shows some intervals whose ratios include a power of 2.

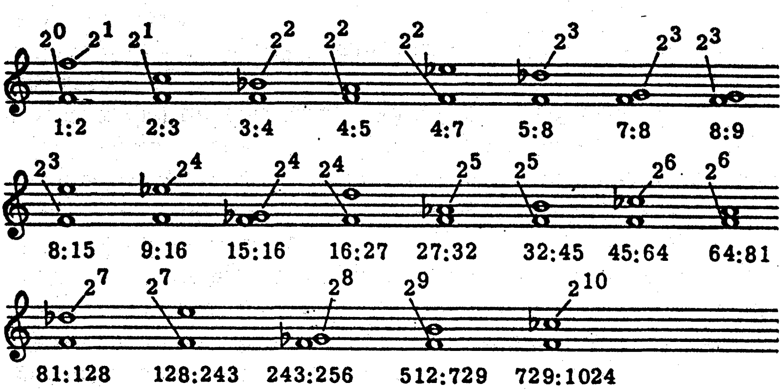


Figure 7. Intervals Which Possess a Power of 2

The presence of a power of 2 in the set of integers is a necessary but not sufficient condition for a chord's possessing a tone which will be perceived as being a root. It is not sufficient, for if the power of 2 is large enough, the fact that it is a root will not be noticed, since it occurs too far from the fundamental of its overtone series. Thus, the diminished fifth is never heard as having a root, although it can be represented by 32:45, and 32 is 2^5. Therefore a coefficient of root strength must be set up. That the root strength diminishes as the power of 2 increases is not arbitrary, but there is some choice between possible scales of diminution of strength of root. I have chosen one which agrees with my hearing and thinking and which, if valid, leads to some interesting consequences. It is in part an empirical matter, and someone else might choose another measure.

The following is the scale of diminution of root strength which will be used here:

Root	Root strength coefficient		
2^0	$1/2^{0^{0}}$	equals $1/1$	equals 1
2^1	$1/2^{1^{1}}$	equals $1/2$	equals .5
2^2	$1/2^{2^{2}}$	equals $1/16$	equals .0625
2^3	$1/2^{3^{3}}$	equals $1/134217728$	equals .0000000008

In general, the root strength of 2^n equals $\tfrac{1}{2}^{n^{n}}$. A root of 2^3, or of 2^n where n is greater than 2, is so weak that for all practical purposes it can be ignored. For convenience we shall multiply this scale by 100. Thus the root strength coefficient of the octave is 100, that of the perfect fifth is 50, that of the major third is 6.25. Thus, tones which can be represented by 2°, 2^1, or 2^2 will be regarded as perceptible roots and all others will not.

According to this scale the major second 8:9 has no perceptible root; neither does the minor sixth 5:8. The fact that the minor sixth sounds as though it has a root may possibly be because the second partial of the lower tone is so strong that it makes the interval of a 4:5 major third with the upper tone, thus conferring a secondary root upon the latter of the value 2^2. This is not true for the major second.

We can now distinguish four different types of sets of tones: (1) sets which have no root, for example the minor third, 5:6, the diminished triad 5:6:7 or 25:30:35, and (2) sets which have one tone as the only root, for example the major triad 4:5:6, the dominant seventh chord 4:5:6:7 or 20:25:30:36, the dominant minor-ninth chord 8:10:12:14:17 or 100:125:150:180:216, (3) sets which have more than one root, of which a single one predominates, for example the minor triad 10:12:15, and (4) sets which have more than one root with no single root predominating, for example, the augmented triad 16:20:25.

Sets which have no root will never have one no matter how they may be inverted. Sets which have only one root will always have the same root in any inversion; their roots are invariant under octave transposition of any of the tones. Sets which have more that one root do not have this property, for octave transposition alters the power of 2 and thus may strengthen one root relative to the others.

The procedure for discovering the root of any set of tones is as follows:

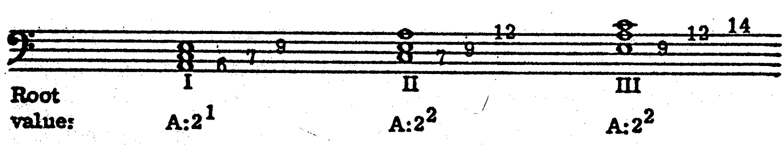
1. Render the set into integers. In some cases this is difficult to do, for each additional tone may alter the interpretation which the mind makes of the stimulus. However, there are cases where it is easy.

2. Consider all inversions of the set. For each inversion consider all two-membered subsets, tabulating the roots and the strengths of the roots. Next consider all three-membered subsets and do the same. Continue this procedure up to and including the set as a whole.

3. Tabulate all the single-rooted subsets. An inversion will be said to be single rooted if one tone is the root of more and stronger intervals than any their tone. If there are none, then the set as a whole is not single-rooted. If one tone is the root of more single-rooted sets than any other tone, then that tone is the predominant root of the set as a whole.

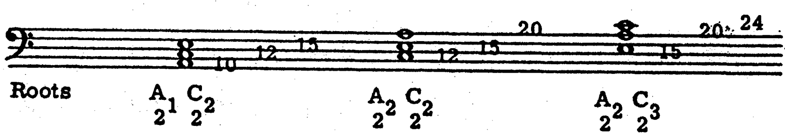
As an example we shall carry out the procedure on the minor triad. Since it has two possible numerical renderings, 6:7:9 and 10:12:15, the results for one will not be decisive; however, in the event that the results for both are the same, then they will be decisive.

First consider the 6:7:9 minor triad. The three possible inversions are shown below:



The first arrangement contains 6:9, 6:7, and 7:9 as two-membered subsets. Of these, only 6:9, which equals 2:3, has a root, and that root has the value 2^1. The three-membered set itself contains no power of 2. Thus this arrangement is single-rooted and A is the root. The second arrangement contains 7:12, 7:9, and 9:12 as two-membered subsets. Of these, only 9:12, which equals 3:4, has a root, and that root is 2^2, which is weaker than the root of the triad in the first arrangement. The three-membered set has no power of 2. Thus the position is single-rooted with A as the root. In the third position the two-membered subsets are 9:14, 9:12, 12:14. Of these, only 9:12, which equals 3:4, has a root and that root is of the value 2^2. The three-membered set has no power of 2. A is the root of this arrangement, as well as of the other two; therefore A is clearly the root of the set, independent of its inversion.

In carrying out the same procedure for the 10:12:15 minor triad the results are as follows:



A is the predominant root, for, although the second arrangement has no predominant root, still, the other two arrangements have A for root. Thus A is the root of the A minor triad no matter which integral representation is preferred.

It is now possible to distinguish between sets of tones which constitute chords and sets of tones which do not. Unless such a distinction is made it is not meaningful to speak about nonchord tones.

A chord can be said to be a set of tones which, when sounded simultaneously, or consecutively, appears to constitute a model for, or variant of, the overtone series. A chord then, will be compounded of tones in a manner similar to that in which the tones themselves are compounded of partials. There are two properties by virtue of which sets of tones will appear to constitute such a model: consonance, and the possession of a single predominant root analogous to the fundamental. The possession of one of these properties does not necessarily ensure the possession of the other, although they are not entirely independent. A certain amount of consonance is necessary in order that the root be perceived at all, because too much dissonance obscures the perception of the compound wave pattern.

Some sets are relatively dissonant but single-rooted; for example, the dominant minor-ninth chord. Others, although relatively consonant, may be either multi-rooted or have no root at all; for example, the dominant major-ninth, which is more consonant than the minor-ninth, has three different roots. The fact that the minor-ninth has a strong unambiguous root permits the unusually large dissonance. This suggests a reason why Bach and others preferred the dominant minor-ninth to the dominant major-ninth.

Sets with a single strong root can be more dissonant without losing their integral identity than can rootless, weak-rooted, or multi-rooted sets. The strength of a tone as a predominant root depends upon two things: the quantity of subsets which have the tone as root, and the size of the power of 2 of the root.

The individual tones of single-rooted sets can be more freely distributed than can those of multi-rooted sets, for single-rooted sets retain their root, or reference tone, so to speak, and hence their identity, no matter how the tones are distributed, while multi-rooted sets may lose their identity under inversion and octave transposition of individual tones, for their predominant root may vary under these transformations. This definition of root substantiates the idea of inversion.

A non-chord tone can now be defined qualitatively and independently of how it is treated in music. Any tone which is added to a single-rooted chord and which does not form a rooted interval with any of the tones of the chord is a non-harmonic tone. For example, if G-flat is added to the chord FAC, then it is a non-harmonic tone. If E-flat is added to FAC it is a harmonic tone, for under one numerical representation, 4:7, it forms a rooted interval with the root of the chord. Under another numerical representation, 5:9, it does not form a rooted interval with the existing root and hence is a non-harmonic tone. It has been treated both ways in music, as a non-harmonic and as a harmonic tone. If D is added to FAC it cannot be regarded as a non-harmonic tone for it forms a rooted interval with A, thus changing the essential nature of the chord. Similarly, the ninth of the minor-ninth chord can be regarded as a non-chord tone, while the ninth of the major-ninth chord cannot.

Name	Numbers	Letters	Fundamental	Predominant root	Quantity of rooted intervals	Strength of predominant root
octave	1:2	FF	F	F	1	100
P. 5th	2:3	FC	F	F	1	50
P. 4th	3:4	FB♭	B♭	B♭	1	6.25
M. 3rd	4:5	FA	F	F	1	6.25
m. 7th	4:7	FE♭	F	F	1	6.25
M. triad	4:5:6	FAC	F	F	2	56.25
m. triad	6:7:9	FA♭C	B♭	F	1	50
m. triad	10:12:15	FA♭C	D♭	F	2	43.75
	12:15:20	A♭CF	D♭	–	2	–
	15:20:24	CFA♭	D♭	F	1	6.25
M.m. 7th	4:5:6:7	FACE♭	F	F	3	62.5
	20:25:30:36	FACE♭	D♭	F	2	56.25
m.m. 7th	10:12:15:18	FA♭CE♭	D♭	A♭	3	6.25
	12:15:18:20	A♭CE♭F	D♭	A♭	3	50
	15:18:20:24	CE♭FA♭	D♭	–	2	–
	9:10:12:15	E♭FA♭C	D♭	F	3	37.5
m.m. 7th	27:32:40:48	FA♭CE♭	A♭	A♭	2	56.25
	16:20:24:27	A♭CE♭F	A♭	A♭	2	56.25
	20:24:27:32	CE♭FA♭	A♭	A♭	1	6.25
	24:27:32:40	E♭FA♭C	A♭	A♭	2	12.50

Figure 8. Root Values

Name	Numbers	Letters	Fundamental	Predominant root	Quantity of rooted intervals	Strength of predominant root
½ D. 7th	25:30:36:45	FA♭C♭E♭	B♭♭	A♭	2	43.75
	30:36:45:50	A♭C♭E♭F	B♭♭	A♭	2	43.75
	36:45:50:60	C♭E♭FA♭	B♭♭	–	2	–
	45:50:60:72	E♭FA♭C♭	B♭♭	A♭	1	6.25
M.M. 7th	8:10:12:15	FACE	F	–	4	–
	10:12:15:16	ACEF	F	A	3	50
	12:15:16:20	CEFA	F	–	4	–
	15:16:20:24	EFAC	F	F	3	50
M. 9th	20:25:30:36:45	FACE♭G	D♭	–	4	–
m. 9th	100:125:150:180:216	FACE♭G♭	B♭♭	F	2	56.25
M. tone cluster	8:9:10	FGA	F	F	1	6.25
quartal chord	9:12:16	FB♭E♭	E♭	–	2	–
quintal chord	4:6:9	FCG	F	F	2	6.25

Figure 8. Root Values (continued)

Thus a chord can be varied by the addition of non-harmonic tones without losing its identity, if the dissonance is not so great as to obliterate the perception of the compound waves as a unit in time.

If the tones of a set occur consecutively in time, the importance of the consonance factor vanishes, for tones which are dissonant when sounded simultaneously do not appear to be so when sounded consecutively. Thus a minor second occurring in a melody does not confuse the listener by its dissonance, while in a simultaneous chord it does. Therefore, the possession of a single predominant root is the principal criterion for determining what sets belong together when sounded consecutively in time, and in the evaluation of scales the consonance factor may be disregarded.

Figure 8, page 36, is a table giving the root values for integral representations of some familiar sets of tones, and (in some cases) their inversions. The first column gives the usual name of the set. The second column gives the integral representation of the set. The third column gives the names of the notes in letters, keeping the lowest tone of the set at F, except in the case of inversions. The fourth column gives the letter name of the fundamental which generates the set. The fifth column gives the predominant root, if any. This is arrived at by allowing equal-valued roots on different tones of the set to cancel each other out. The sixth column gives the quantity of intervals of the set which possess a root with a strength of 2^2 or stronger. The seventh column gives the strength coefficient of the predominant root.

From the table we see that the only single-rooted sets investigated are the fifth, fourth, major third, minor seventh, major triad, minor triad (in its integral representation using the seventh partial), dominant seventh chord in any representation, dominant minor-ninth chord in any representation, and the cluster of three major seconds. The fact that all of these chords except the cluster have been widely used may be explained by the peculiarly ambivalent character of the major second, there being three different candidates for numerical representation of each second. Another unfortunate property which the major cluster (8:9:10) has is an excessive quantity of clashing partials of the tones of the set, due to their close spacing.

The fact that the 10:12:15 minor triad has as its fundamental the tone a major third below its predominant root, may explain the effectiveness of the sixth scale step in a minor key.

The only single-rooted sets without non-chord tones whose roots are 2^2 or stronger, are the octave, perfect fifth, perfect fourth, major third, minor seventh, major triad, and the dominant seventh chord.

Summary

1. The root of a set of tones, (or that tone which appears to be the most important) is the tone of the set which is represented by the lowest power of 2, and is, therefore, either the fundamental of the set or an octave transposition of that fundamental.

2. Tones having a power of 2 greater than 2 will not be heard as being roots, for their relation to the fundamental will be too remote to be perceived.

3. The following types of sets exist: (1) sets with no roots, (2) sets with one root only, (3) sets with more than one root, of which one predominates, and (4) sets with more than one root of which none predominates.

4. A chord is a set of tones which, when sounded simultaneously, appears to constitute a model for, or variant of, the overtone series. It possesses a single predominant root which corresponds to the fundamental of the series.

5. A non-chord tone is a tone added to a single-rooted set which does not form a rooted interval with any tone of the set.

6. The only single-rooted sets whose roots are 2^2 are stronger, and which contain no non-chord tones are the octave, perfect fifth, perfect fourth, major third, minor seventh, major triad, and the dominant seventh chord.

Chapter Four

The Diatonic Set

THE METHOD of determining the root of a set which was established in Chapter III can also be applied to the larger sets of tones usually called scales, or keys.

The just diatonic set occurs in seven different arrangements, similar to the different inversions of a chord, within an octave range, as shown in figure 9, page 42. To find the root of this set we need consider only the two-membered subsets and the three-membered subsets, for the subsets with more members than three contain no roots of the value 2^2 or stronger, and thus may be disregarded.

The first arrangement of tones, which has F as the lowest tone, is not single-rooted, for of all the two- and three-membered subsets those built on F and G have the strongest roots, and they are equally strong. The second arrangement is single-rooted and the root is G. The third arrangement, as well as the fourth and fifth arrangements, are single-rooted and their root is C. The sixth arrangement has F as the single root, while the seventh, like the first, is dual-rooted, F and G being equally strong roots. Thus only five of the possible seven positions are single-rooted, and of these, three have C as the root, one has G as the root, and one has F. Thus C is the predominant root of the set as a whole.

We notice that the fifth arrangement, in which C is the lowest tone, is the most consonant arrangement, for it occurs lower in its own overtone series than do any of the other arrangements. The C major diatonic set is therefore a set in which the most consonant position is also root position, (a property it shares with the minor triad, and the dominant seventh chord, but not the major triad). The positions of the diatonic set are similar to the inversions of a chord, and we can speak of root position of the C major set and of the first to sixth inversions, just as we speak of root positions and the inversions of the C major triad.

The difference of root value and degree of consonance in the different inversions of the diatonic set serves to explain the existence of the church modes, for the church modes correspond to the different inversions of the diatonic set.

If a different root-strength scale were used the results would be similar. But it must be pointed out that if no distinction were made between stronger and weaker roots, then F, the generating tone of the Pythagorean diatonic set, would be the root of the diatonic set, for F is the root of the greater quantity of intervals, while C is the predominant root of the greater quantity of inversions.

This duality makes possible the symbolic richness of motion in different directions from the root, or tonic, C. Melodic motion of tones or roots of chords from C to F is down, in, back toward the past, the generating tone, while melodic motion of tones or roots of chords from C to G is up, out, forward.

The situation is different with the Pythagorean diatonic set: CDEFGAB = 384:432:486:512:576:648:729 due to the fact that the Pythogorean third is 64:81, or $2^6 : 3^4$, and 2^6 falls far beyond the range of perceptible roots. The Pythagorean set has no three-membered subsets with roots sufficiently strong to be worth considering. It has no single-rooted arrangement. However, if we consider the sets of roots of the different arrangements, the most consonant one is the set of roots for the arrangement which has A as the lowest tone, for its two roots are A and C, while all the other positions have two or three roots at an interval of a major second. This, plus the fact that the Pythagorean minor third is more consonant than the Pythagorean major third, may account for the fact that A, the first letter of the alphabet, was *mese* to the Greeks. It was considered by Aristotle to be the most important tone of the scale:

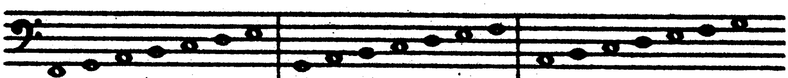
> In all good music *mese* occurs frequently, and, if they leave it, they soon return to it, as they do to no other note.[1]

The arrangement which has C as the lowest tone is the most consonant arrangement of the Pythagorean set, as it is of the just diatonic set.

The root distribution of the just and Pythagorean diatonic sets is shown in Figure 9, page 42.

[1] Aristotle, *Problems,* Book XIX.

I II III



I — 32:36:40:45:48:54:60

	F	G	A	B	C	D	E
2^1	1	1	1				
2^2	1	1			2		1
2^3	2		1		1		
2^4	1				1		
2^5	1						
2^6							

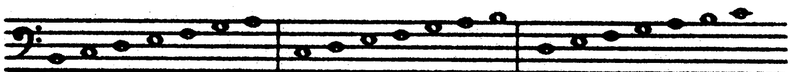
II — 36:40:45:48:54:60:64

	F	G	A	B	C	D	E
2^1		1	1				
2^2	1	1			2		1
2^3	1		1		1		
2^4	2				1		
2^5	1						
2^6	1						

III — 40:45:48:54:60:64:72

	F	G	A	B	C	D	E
2^1			1		1		
2^2	1	1			1		1
2^3	2	1	1		1		
2^4	1				1		
2^5	1						
2^6	1						

IV V VI



IV — 45:48:54:60:64:72:80

	F	G	A	B	C	D	E
2^1					1		
2^2	2	1	1		1		1
2^3	1	1			1		
2^4	1		1		1		
2^5	1						
2^6	1						

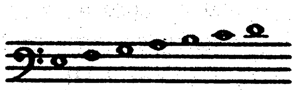
V — 24:27:30:32:36:40:45

	F	G	A	B	C	D	E
2^1					1		1
2^2	2	2	1		1		
2^3	1		1		2		
2^4	1						
2^5	2						
2^6							

VI — 27:30:32:36:40:45:48

	F	G	A	B	C	D	E
2^1	1						1
2^2	1	2	1		1		
2^3	1		1		1		
2^4	1				2		
2^5	2						
2^6							

VII

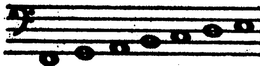


VII — 30:32:36:40:45:48:54

	F	G	A	B	C	D	E
2^1	1	1					1
2^2	1	1	1		1		
2^3	1		1		2		
2^4	2				1		
2^5	1						
2^6							

Figure 9. Root Distribution in the Just Diatonic Set

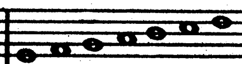
I



512:576:648:729:768:864:972

	F	G	A	B	C	D	E
2^1	1	1	1				
2^2					1	1	1
2^3	1	1	1		1	1	
2^4	1	1					
2^5					1	1	
2^6	1	1			1		
2^7	1						
2^8					1		
2^9	1						
2^{10}							

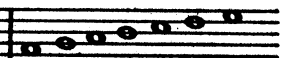
II



576:648:729:768:864:972:1024

	F	G	A	B	C	D	E
2^1		1	1				
2^2	1				1	1	1
2^3		1	1		1	1	
2^4	1	1					
2^5	1				1	1	
2^6		1			1		
2^7	1						
2^8	1				1		
2^9							
2^{10}	1						

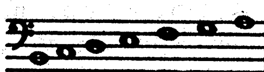
III



648:729:768:864:972:1024:1152

	F	G	A	B	C	D	E
2^1			1		1		
2^2	1	1				1	1
2^3	1		1		1	1	
2^4		1					
2^5	1	1			1	1	
2^6					1		
2^7	1	1					
2^8	1				1		
2^9							
2^{10}	1						

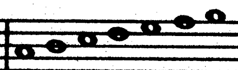
IV



729:768:864:972:1024:1152:1296

	F	G	A	B	C	D	E
2^1					1	1	
2^2	1	1	1				1
2^3	1	1			1	1	
2^4			1		1		
2^5	1	1				1	
2^6	1				1		
2^7		1					
2^8	1				1		
2^9							
2^{10}	1						

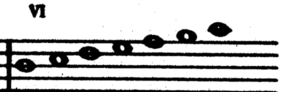
V



384:432:483:512:576:648:729

	F	G	A	B	C	D	E
2^1					1	1	1
2^2	1	1	1				
2^3	1	1	1		1	1	
2^4					1	1	
2^5	1	1					
2^6	1	1			1		
2^7					1		
2^8	1						
2^9	1						
2^{10}							

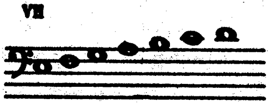
VI



432:483:512:567:648:729:763

	F	G	A	B	C	D	E
2^1	1					1	1
2^2		1	1		1		
2^3	1	1	1			1	
2^4					1	1	
2^5	1	1			1		
2^6	1	1					
2^7					1		
2^8	1				1		
2^9	1						
2^{10}							

VII



483:512:576:648:729:768:864

	F	G	A	B	C	D	E
2^1	1	1					1
2^2			1		1	1	
2^3	1	1	1		1		
2^4	1					1	
2^5		1			1	1	
2^6	1	1					
2^7					1		
2^8	1				1		
2^9	1						
2^{10}							

Figure 9. (Cont.) Root Distribution in the Pythagorean Diatonic Set

By a similar procedure it can be shown that the pentatonic set F, G, A, C, D has a root in just intonation, and that the root is F. In Pythagorean intonation, the pentatonic set has no single-rooted arrangements, as is the case in the Pythagorean intonation of the diatonic set; but if the sets of roots are considered, it is the DF set, which occurs in the arrangement having D as the lowest tone, which is the most consonant arrangement. In the event that it were to be argued that since the pentatonic set is a subset of the diatonic set, and therefore its having F as root strengthens F, as against C, in the diatonic set, it must be pointed out that the diatonic set includes a pentatonic set on C as well as on F.

I conclude that the diatonic set produces in the hearer a hierarchy of tones containing a most important tone (call it root or tonic), and that this root, or tonic, is not a matter of convention, for it stems from the inherent properties of the tones of the set and of the human listener. This root serves as a common measure, a reference tone, to which the other tones are consciously or unconsciously related by the listener.

Summary

1. The seven tone diatonic major set on C in just intonation has C as its predominant root.
2. The church modes are to the diatonic major set as inversions are to chords.
3. A root is a common measure, or reference tone.

Chapter Five

The Twelve-Tone Set

NO TWELVE-TONE set has a predominant root. In order to demonstrate this it is necessary to know what twelve-tone sets exist, either in practice or conceptually. By "exist conceptually" I mean: follow from a well-defined method of tone generation which contains a non-arbitrary statement of cut-off, or when to stop. The equal-tempered set is well-defined, although its ratios are incommensurable. The Pythagorean twelve-tone set is well defined. But there is no well-defined just twelve-tone set.

A just set makes use of 1:2 octaves, 2:3 fifths, and 4:5 thirds in order to arrive at its pitches, the frequencies of which depend upon the order in which they are derived. What non-arbitrary principle is to prescribe when a fifth is to be used and when a third? No such principle has ever been stated.

The fact that the diatonic set has a predominant root, and thus constitutes a gestalt in itself, suggests a principle of generation of a just twelve-tone set which is less arbitrary, at least, than any previous principle. It is the following:

Erect on each tone of a just diatonic set a set identical to it. This procedure will result in the addition of the tones F-sharp, C-sharp, D-sharp, G-sharp, and A-sharp, which are markedly different from the existing tones, and also a host of other tones which are just a little different from the existing tones. In order to eliminate these slightly different tones from the set we can choose for each tone either the smaller value, or the value which occurs first. If we choose the value which occurs first, then an order of construction must be specified, and that is possible to do by the Pythagorean method of successive fifths. However, this decision is somewhat arbitrary, although well defined, for it assumes a preference, which cannot be justified easily, for the Pythagorean set over the just set.

The values for these sets are the following:

F	F#	C	C#	G	G#	D	D#	A	A#	E	B	
$\frac{1}{1}$	$\frac{135}{128}$	$\frac{3}{2}$	$\frac{270}{256}$	$\frac{9}{8}$	$\frac{1215}{1024}$	$\frac{27}{16}$	$\frac{225}{128}$	$\frac{5}{4}$	$\frac{675}{512}$	$\frac{15}{8}$	$\frac{45}{32}$	first occurred
$\frac{1}{1}$	$\frac{135}{128}$	$\frac{3}{2}$	$\frac{25}{16}$	$\frac{9}{8}$	$\frac{75}{64}$	$\frac{27}{16}$	$\frac{225}{128}$	$\frac{5}{4}$	$\frac{675}{512}$	$\frac{15}{8}$	$\frac{45}{32}$	smallest

Even if we were to accept one of these sets as being non-arbitrary, which seems a dubious procedure, still, all possible contenders, arbitrary or non-arbitrary, for a just twelve-tone set, allow a close approximation to a major triad to be built on all tones of the sets, thus rendering them centerless. Any difference in quantity and quality of roots occurs well below the level of perceptible root strength (if my scale of root strength is accepted), while among the perceptible roots the orientation of each tone is the same as any other, with the exception of C-sharp and A-sharp. These two tones require an E-sharp which is not in the set in order to build a major triad on them.

The Pythagorean twelve-tone set has no root, for each of its inversions is identical with all the others with regard to perceptible roots. The equal-tempered set has no root, for none of its subsets is commensurable.

Summary

1. The only non-arbitrary well-defined twelve-tone sets which have been considered by theorists are the Pythagorean twelve-tone set and the equal-tempered twelve-tone set.
2. Neither of these has a predominant root.

Chapter Six

The Nineteen-Tone Set

THERE ARE an infinite number of ways to create sets of tones larger than the twelve-tone set, for the octave can be divided arbitrarily into as many different tones as may be desired. But are there any non-arbitrary sets having more than twelve tones? It has been maintained that the nineteen-tone set is such a set, an organic development growing out of, but not destroying, the pentatonic and diatonic sets.[1]

The argument will be made here that this is not the case; on the contrary, the nineteen tone set is simply an arbitrary division of the octave into nineteen equal parts bearing no more organic relationship to the diatonic and pentatonic sets than would the division of the octave into twenty, twenty-one, twenty-two, etc., equal parts.

One way in which a set of tones can be considered to be an outgrowth of the diatonic and pentatonic sets is by carrying the principle by which these sets were derived even farther than it was carried in deriving the original sets. The nineteen-tone set is not the result of such a process; for the principle of tone derivation of the Pythagorean system, of which the nineteen-tone set purports to be an extension, is the following:

2:3 fifths are superimposed until a tone is reached which comes closer to coinciding with a tone derived by successive superimposition of 1:2 octaves in the same direction, than any of the preceding tones. Figure 10 presents a logarithmic picture of this process.

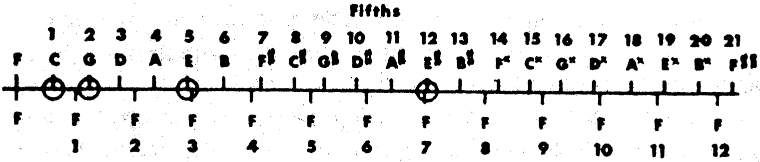


Figure 10. Logarithmic Picture of the (extended) Pythagorean Scale

[1] Yasser, Joseph, *A Theory of Evolving Tonality,* American Library of Musicology, N.Y., 1932.

Those tones are circled which are closer to an octave transposition of the generating tone than any tone which precedes them. E is such a tone, and E-sharp is also such a tone, but E-double-sharp, the nineteenth fifth, is not such a tone. It can be shown by continued fractions that the next such tone is the forty-first fifth.

This fifth-derived tone which is closer to an octave-derived tone than any of its predecessors is then declared identical to the tone it is close to, which means that the small difference-interval is considered to be beyond the perception threshold and is thus ignored.

The pentatonic set is justified by this process, and so is the equal tempered twelve-tone set, but the nineteen-tone set is not, for the interval between E-double-sharp and F is not smaller than any preceding interval. In fact, it is so large that two other previously derived tones occur between it and the octave transposition of the generating tone, as shown in Figure 11.

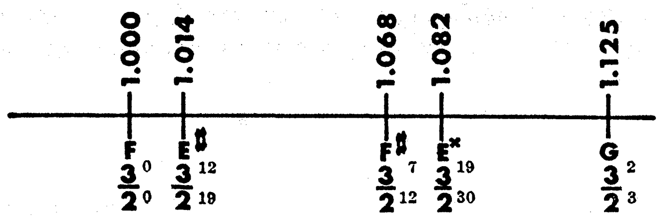


Figure 11. Tones Close to F

Therefore, if E-double-sharp were to be declared identical with F, so also should these other two intervening tones.

Another way in which a set of tones could be considered to be an outgrowth of the diatonic and pentatonic sets might be that it improves the tuning of these sets by the addition of tones which more closely approximate the paradigm intervals of the set. This is not the case with the nineteen-tone equal-tempered set, for the fifth, the generating interval of the whole Pythagorean system, is five cents farther from the pure fifth than is the fifth of the equal-tempered twelve-tone set. The following is a comparison of fifths and major thirds in cents:

	pure	twelve-tone equal-tempered	nineteen-tone equal-tempered
fifth:	702	700	693
major third:	386	400	379

The nineteen-tone equal-tempered set increases the accuracy of the major third at the expense of decreasing the accuracy of a more fundamental interval.

The argument has been made in favor of the nineteen-tone set that it brings in a new smallest interval without losing the old intervals, which enriches the system heretofore limited by the human perception threshold, and that the man of the future will be able to differentiate it from the other intervals, just as the diatonic set introduces the minor second as a refinement over the major second, the smallest interval of the pentatonic set.

This new smallest paradigm interval introduced by the nineteen-tone set is the Pythagorean comma, A-flat to G-sharp, which is a negative quantity when considered with reference to the diatonic sets. A negative interval can exist only in the imagination, not on a keyboard. No matter how fine the hearing of the man of the future might be, he never could hear this. Were it to be argued that the negativity of this interval is only the result of diatonic language, the reply is that the diatonic language was constructed to fit the diatonic set; the fact that sharps and flats are in the language results from the fact that the diatonic set forms a gestalt in itself by virtue of its single-rootedness. If the nineteen-tone set is not considered to have grown organically out of the diatonic set, then the Pythagorean comma may indeed be a new smallest positive interval, the existence of which in the nineteen-tone set destroys any organic connection between it and the diatonic and pentatonic sets.

The nineteen-tone equal-tempered set, like any other equal-tempered set, has no root, since each tone is oriented to all the other tones in exactly the same way. Since it is not an organic development of the pentatonic and diatonic systems, neither does it have a root by virtue of historical association with those rooted sets.

Summary

1. The nineteen-tone set is not an organic development out of the pentatonic and diatonic sets, but an arbitrary division of the octave into nineteen equal intervals.

2. The nineteen-tone set has no predominant root either by virtue of its construction or of its history.

Chapter Seven

Mode And Key

SINCE THE diatonic set has a single predominant root it is possible to clarify the meaning of the word "mode", and thus to distinguish between mode and key in music which is not restricted to the range of an octave. "Key" will be defined as the cardinal diatonic set, and "mode" as the ordinal diatonic set. Thus a key consists of a diatonic set of tones in any order of occurrence, while a mode is a diatonic set of tones, ordered in time in such a way as to emphasize a particular tone of the set. To each key, then, there correspond seven different modes.

The various means of establishing this order of importance of tones are numerous. They reach from crude emphasis to the subtler and far more effective means of combining tones in a manner analogous to the inner structure of the single tone.

A tone can be emphasized by deliberate accent, orchestration, frequency of occurrence, duration, melodic prominence as peak-tone, or tone skipped to or away from, metrical position, the use of its leading-tone, trilling, and ornamentation. But the strongest and most effective way to emphasize a tone, a way which has been lost in much contemporary music, is emphasis by the use of sets of tones which have the given tone as root. These sets may be composed of tones which are members of single-rooted chords, or they may be composed of roots or tonics themselves, as in the case of the V-I harmonic progression.

Emphasis on a tone which is the result of that tone's being the root of a set will be called intrinsic emphasis. All other emphasis will be called extrinsic.

When the intrinsic and extrinsic factors emphasize the same tone, then the hierarchy of tones will be the clearest. Complete clarity, however, is not always desirable; it varies in a composition as a whole as well as from composition to composition. The minor triad is not as completely clear as is the major triad, since in one numerical representation it has two roots; perhaps some of its effectiveness stems from this fact.

When a diatonic set is used in such a way that a tone other than its root is consistently emphasized, then we have what is often called modal music. This can be achieved by range restriction, as in the case of Gregorian chant, or, when range restriction is eliminated, by employing the above means of emphasis.

Altered tones change the set itself, and thus the mode as well. It is possible to write music in all seven modes of a key without the alteration of a single tone. The relative difficulty of establishing each mode can be seen in Figure 9, page 42. The tones with the greatest quantity of roots are the tones on which modes can most easily be established. The most difficult mode to establish is the one emphasizing the seventh scale step of the diatonic set, because the most effective means of emphasis is not available to it, since it is not the root of any subset. Therefore, if it is to be established at all, it must be by the more brutal, obvious, and less effective means of loudness, frequency, duration, etc., as well as avoidance of the use of rooted intervals on other tones. This accounts for its virtual absence from the musical literature before the twentieth century.

There are, then, four different possibilities of variation of diatonic sets within the tonal framework of the twelve-tone equal-tempered set:

1. The cardinal set, or key, may be changed, while the mode remains the same. For example, changing from C major to G major.

2. The ordinal set, or mode, may be changed, while the cardinal set, or key, remains the same. For example, changing from C major (Ionian mode of C) to the Aeolian mode of C, which emphasizes A.

3. The cardinal set, or key, and the ordinal set, or mode, may be changed, while the tone which is emphasized remains the same. For example, changing from C major (Ionian) to the Aeolian mode of E-flat which emphasizes C.

4. The key, the mode, and the emphasized tone may all change, while nothing remains constant. For example, changing from C major to the Aeolian mode of B which emphasizes G-sharp.

Inner variations of the diatonic set can also be made. The harmonic minor set is such a variation. It is not the same as the Aeolian mode, despite the fact that it grew out of it, and has a similar character. By the single operation of substituting G-sharp for G in the Aeolian mode of what can now be called the key of C, two ends are achieved: A is given a leading-tone, which serves to emphasize A, and C, the previous root of the set, is deprived of its fifth, G, which was the principal means of emphasizing C. Thus the substitution of G-sharp for G in the C major set, changes the set and thus also the center of the set.

By applying the method of root determination used in Chapter IV to the harmonic minor set, we find that A is its predominant root, with E and F the next most important roots.

Consider all the sets which occur as a result of emphasizing the principal note of the modes by means of a lower leading-tone. The resulting sets are the following:

Dorian:	D	E	F	G	A	B	C-sharp
Phrygian:	E	F	G	A	B	C	D-sharp
Lydian:	F	G	A	B	C	D	E
Mixolydian:	G	A	B	C	D	E	F-sharp
Aeolian:	A	B	C	D	E	F	G-sharp
Locrian:	B	C	D	E	F	G	A-sharp
Ionian:	C	D	E	F	G	A	B

The Ionian, Aeolian, Lydian, and Mixolydian have already been considered implicitly: A leading-tone is already present for F and C; giving G a leading-tone does not change the inner nature of the set but only transposes it. The predominant roots of the Aeolian and Ionian sets have been shown to be A and C.

The other three sets however, are genuinely different from each other and require separate consideration. When the roots from these sets are extracted a reason why the Aeolian and Ionian modes took precedence over all the other modes during the seventeenth, eighteenth, and nineteenth centuries appears. It is that only these sets have single predominant roots, with the tone indicated by the mode as root. The sets derived by raising the leading-tone of D, E, and B have no predominant root, and the roots of the different single-rooted inversions do not occur on the principal tone of the mode. The following chart gives the root distribution of the different sets:

	Roots of single-rooted inversions						
Step of mode in bass:	1	2	3	4	5	6	7
Dorian	E	G	–	–	A	–	–
Phrygian	F	F	A	C	C	C	F
Aeolian	A	–	E	–	–	F	A
Locrian	C	C	–	G	G	G	C

Figure 12. Root Distributions of Sets Derived from Modes

Investigation of the roots of other sets having more than seven tones shows that the addition of F-sharp to the diatonic set does not completely destroy its hierarchical properties, although it does weaken them. C is still the root of this set, but not so strongly as it is the root of the seven-tone set. Any further additions of sharps to the diatonic sets results in the obliteration of the single predominant root. There is little historical application of the set containing F and F-sharp, for in most music the introduction of F-sharp takes place as a substitution for F-natural, rather than as an addition to the set. In music of the later nineteenth and twentieth centuries in which this is not the case, not only is F-sharp added to the set but so also are all the other accidentals, thus rendering the twelve-tone set, rather than the eight-tone set including F-sharp, the basis for the music.

Summary

1. "Key" is defined as the diatonic set of tones, independent of their order of occurrence. It is thus a cardinal set of tones. Mode is defined as the ordinal set, for it is a function not only of the tones used, but of the order of their occurrence in time. For each seven-tone cardinal set or key there exist seven different ordinal sets or modes.

2. There are four qualitatively different types of changes between key and mode.

3. The alteration of the fifth scale step of the diatonic key to provide a leading tone for the sixth scale step changes the set from major to harmonic minor. This is the only set derived in a similar manner, which has a single predominant root. This explains the importance of the major and minor keys.

Chapter Eight

Analysis

ROMAN NUMERAL analysis of music is foundering on the following paradox: "The altered tone is in the key; and the altered tone is not in the key." The very existence of the Roman numeral as a meaningful symbol depends upon the exclusion of altered tones from the key. Thus it is a violation of the language to speak of a "II sharp 4" chord, for if the 4 is sharped, then the chord is no longer a II chord. As we shall see, however, it has a metaphorical justification.

Every good metaphor can be translated into a non-metaphorical statement which is literally true. If people fail to do this, the metaphorical truth will be cast out along with the literal falsity, as the Roman numeral is being cast out of musical analysis to the detriment of that analysis.

The fact that the diatonic set (the key without altered tones) has a root, while the twelve-tone set (the key with altered tones) does not, enables us to translate the metaphor implicit in the above contradiction, whose literal falsity must be accepted if meaningful language is to survive.

Since the diatonic set is single-rooted it forms a hierarchical entity in itself, each of whose members has a particular position in the hierarchy. This position in the hierarchy we shall call the "function" of the tone, and assign a number to it from 1 to 7, as has been done traditionally. The triads built upon these functional scale steps will be numbered by Roman numerals in order to distinguish them from the Arabic numerals indicating the single tone, as has also been done traditionally. A tone, then, (or a single-rooted chord with that tone as a root), if it belongs to a single-rooted set, has a particular function in that set. "Function" has usually been used with reference to the diatonic major and minor sets. We see that in this generalized sense of the word its meaning might extend to the hierarchical position of chord members relative to the root of the chord. It might even be extended to include the position of the partials relative to their fundamental.

The Roman numeral has in the past been used in two different senses by analysts: (1) It has been used merely as a label to differentiate chords from each other, and (2) it has been used as a name for a function. When it is used in the first sense the altered tone is perfectly acceptable, for we can label chords relative to any reference point, as long as we don't change the reference point without mentioning the fact. Thus one can interpret any set of three tones as constituting any Roman numeral of any key, when the Roman numeral is used purely nominally, as a label. Such Roman numeral labels can be applied to a twelve-tone composition, or to any composition at all; for by means of enharmonic equivalence it is possible to spell any three tones as a triad. But if this is done, it is only an exercise in the application of an arbitrary terminology and yields no information whatever about the music to which it is applied.

If, on the other hand, the Roman numeral is used as a name for a function, its use reveals regularities which occur in the music of some historical periods and which do not occur in the music of other historical periods. The discovery of such regularities or absence of such regularities is the chief purpose of the analysis of music. Therefore, the Roman numeral as a functional label is a valuable tool for the music theorist.

The Roman numeral can exist as a name for a function only if altered tones are not recognized as being in the key. Possibly in the attempt to demonstrate non-existent similarities between contemporary and classical music, the Roman numeral has been applied functionally to music to which it is only applicable as a label. But it should either be used as a function or not at all, because if it is not used as a function it can be applied equally to any combination of musical tones, and thus have no purpose whatsoever.

Since the diatonic set is a real entity, a gestalt in itself, the whole of which is greater than the sum of its parts, such sets can be regarded as units, and combined, as chords can be combined, into higher order sets which may still retain the same root. Thus a diatonic set on C can be followed by a diatonic set on G, and the sum of the two sets will retain the root C, just as the combination of the major triads on C and on G retains C as the root. Other combinations of diatonic sets, similar to combinations of single-rooted triads, can be made as follows: Sets on C, F, and G can be formed analogous to the structure of the diatonic set itself as a combination of major triads in C, F, and G. Diatonic major sets on C, F, and G can be combined with diatonic minor (harmonic minor) sets on D, E, and A to form a structure analogous to the structure of the diatonic set itself as a combination of major triads on C, F, and G and minor triads on D, E, and A. Similarly, diatonic minor sets on C, F, and G can be combined with diatonic major sets on A-flat, B-flat, and E-flat to form a structure analogous to the Aeolian mode of the major set.

Each of these diatonic sets is a function, in the sense we have defined, of the higher order set produced by their combination. Therefore Roman numerals, analogous to the Roman numerals applied to chords, can be applied to keys as well. But these two symbols must be differentiated from each other, for, as will be shown, key progressions do not follow the same patterns as do chord progressions. It is the confusion of these two different meanings of the Roman numeral which has led to the paradox resulting in the abandonment of the Roman numeral. In order to differentiate between Roman numerals which apply to chords and those which shall apply to keys, we shall attach an index to the latter; for example: V′, I′ indicates the progression from the dominant key to the tonic key, while V, I indicates the progression from the dominant chord to the tonic chord, and 5, 1 indicates the melodic progression from the fifth scale-step to the tonic.

The higher-order set will be called a second-order key. This higher-order set itself may or may not be combined into a further higher-order set. If it is, then it in turn becomes a function of the next higher-order set, and will be designated by a Roman numeral followed by two superscripts (″). This procedure can be carried on indefinitely if the music warrants it. Thus the tones C D E F G A B are 1 2 3 4 5 6 7 of the key of C; G B D is V of the key of C, and the set of tones G A B C D E F-sharp is V′ of the second-order key of C.

In compounding sets into a minor second-order key it is a problem whether or not to call the compound a minor second-order key or the Aeolian mode of a major second-order key, for the major dominant occurs only at the level of the key, not at the level of the second-order key. It seems advisable, however, to call it a minor second-order key, even though it is really the Aeolian mode of the relative major second-order key, because a consistency of function can thereby be maintained, due to the fact that the principal chordal functions of a minor key are I, IV, and V, just as they are of a major key.

If we recognize the fact that an altered tone cannot be added to a set without changing the very nature of the set, and hence the distribution of the Roman numerals, then we can proceed to make meaningful tonal analyses of music without recourse to a private mysterious intuitive process.

In order to do this it is necessary to observe set changes when they take place, whether the sets are chords or keys. They do take place increasingly frequently from the sixteenth century to the present.

The graphs on pages 58 to 63 show the distribution of diatonic sets in music of different periods. In the event that a group of tones belongs to several different sets, that set was chosen which lasted the longest. If this stipulation were not made, it would be possible to interpret each different tone as belonging to a different set. The sets are either the diatonic major or the harmonic minor. The only tones apart from these sets whose occurrences do not result in a change of set, are (1) the raised sixth scale-step in minor, when it is next to the raised seventh scale-step, (2) tones which act as lower leading-tones to each tone of the tonic or dominant triads, (3) chromatic runs, or glissandi. When any of these exceptions occurs it is marked by an asterisk.

In the graphs, the line marked A is the graph of the tonics of the keys. A double line occuring at a particular level indicates the relative minor set of the major set indicated by the letters marking the particular level. The line marked B is the graph of the tonics of the second-order keys, if there are any. The line marked C is the graph of the tonics of the third-order keys, if such there be. The Roman numerals, where they exist, of the second-order keys and higher-order keys are given for each graph.

These graphs do not show the root movement of chords, but only the root movement of the larger sets. On each of the horizontal levels further division could be made according to chord progression, and even to the melodic progressions of single tones. Furthermore, these tonal levels could be divided still further according to the distributions and intensities of the partial tones. The sum of these partial tones is the picture which is shown by the

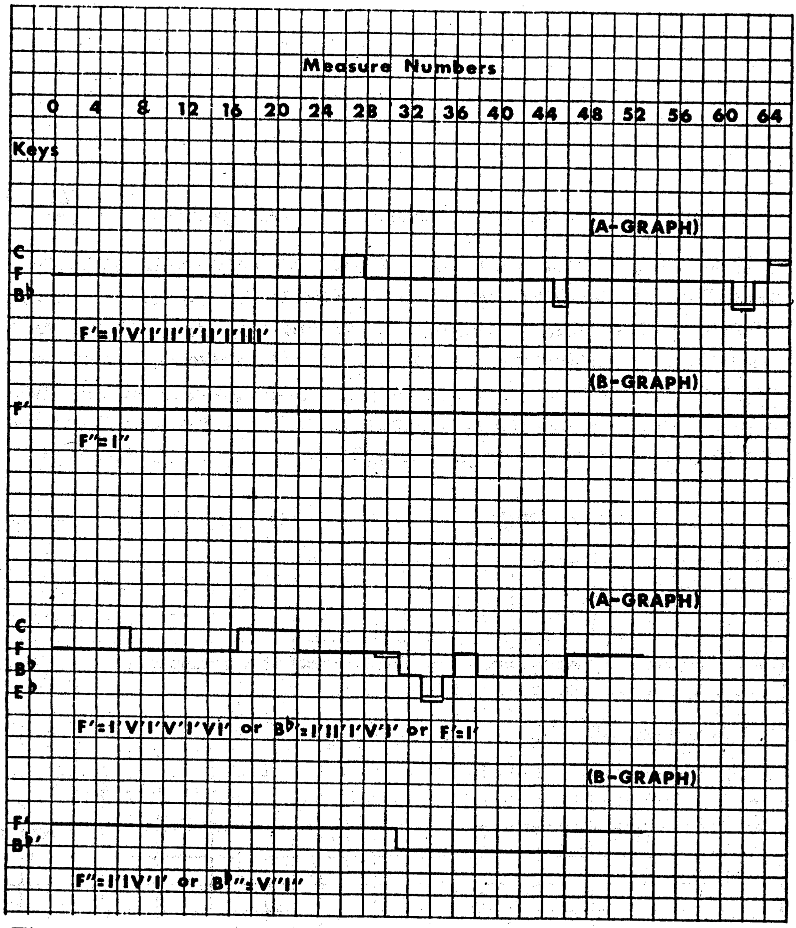


Figure 13. Top: *Lucem Tuam* by John Redford, 1485-1545; Bottom: *Alla Riva del Tebro* by Palestrina

oscillograph when it records sound wave patterns. Thus the way may be opened to a method of abstracting tonal hierarchies from the purely mechanical rendering of sound by an oscillograph.

At the melodic level a further complication appears which is negligible with regard to roots and keys. It is the absolute differences in pitch which become relevant at this level. For in a melody there is a great difference between a tone and its upper octave, while at the level of generality of the root, it can be ignored. At the melodic level we have a new and absolute up and down entering into the picture.

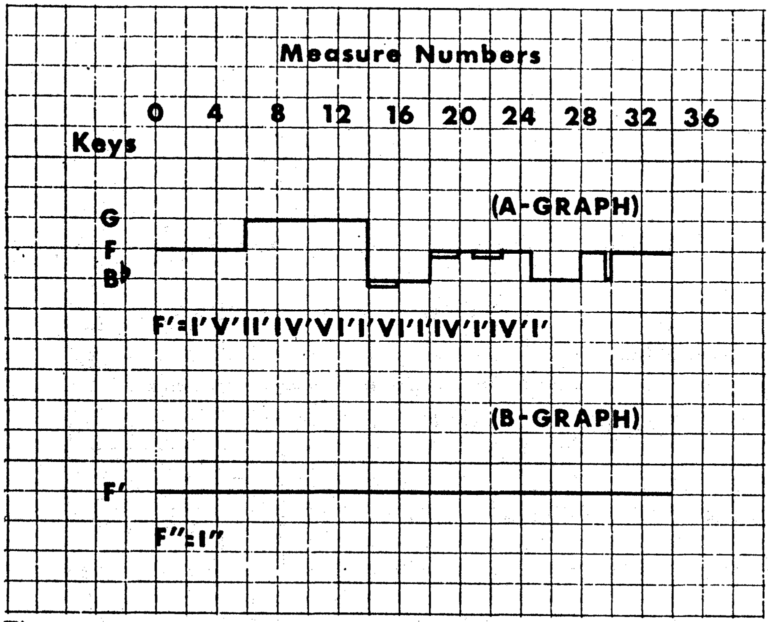


Figure 14. *Two-Part Invention Number 8* by Bach

These graphs are intended to show the larger structure which has too often been ignored in favor of the detailed structure, and to point out the enormous difference between the music preceding the seventeenth century and that following the nineteenth century.

Some important properties which can be determined by such analyses as these are the following:

1. Whether or not the composition can be compounded into a single higher order set. If it can, then it is an example of a composition which unfolds a single tone.
2. The degree of tonal complexity of music, which is measured precisely by the number of superscripts of the Roman numeral of the highest level key which can be compounded. This number is an indication of the quality of simultaneous roots which may exist in a composition. This is a measure of its hierarchical order. It varies inversely as the tonal entropy (to be discussed in Chapter XII).
3. The rate of change of keys.
4. The predominant interval between adjacent keys, if such an interval exists. In the Bach examples this interval is the fifth.
5. The presence or absence of any keys at all.

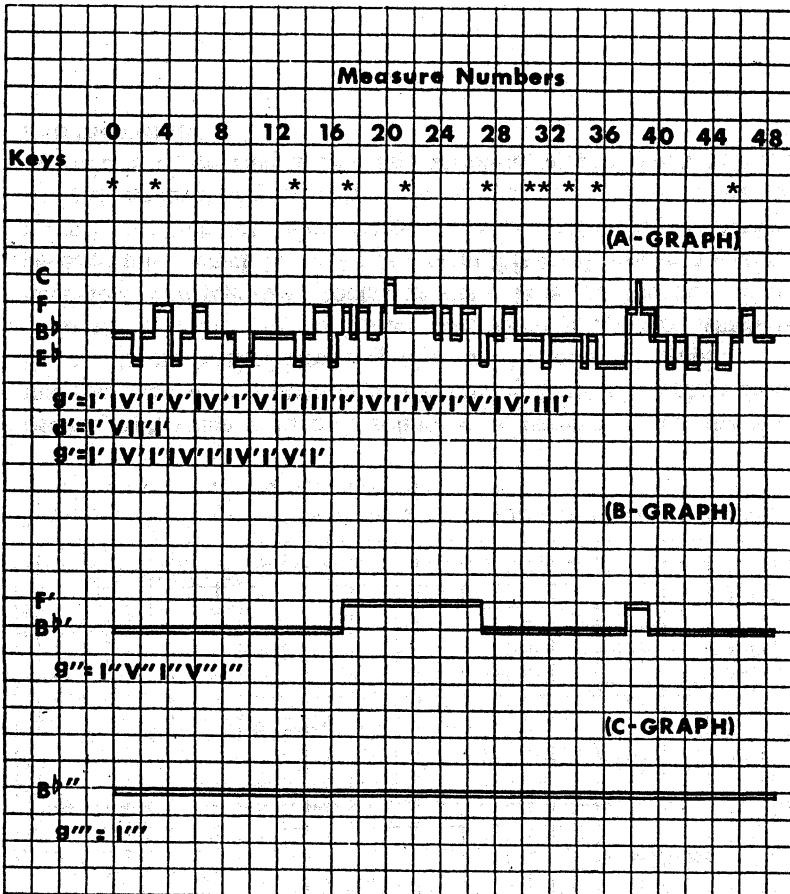


Figure 15. *Agnus Dei,* from the *Mass in B Minor* by Bach

The fact that a key is a single-rooted unit serves to explain the importance of the tritone, an interval which, in isolation, or in a non-diatonic context, seems to be of little value due to its relative dissonance, ambiguity of numerical representation, and rootlessness. But its value, like that of the diminished triad and the diminished seventh chord, comes from another source: as Schenker points out,[1] it is univalent, that is, only one such sound occurs in a

[1] Schenker, Heinrich, *Harmony,* Chicago University Press, Chicago, 1954, p. 127.

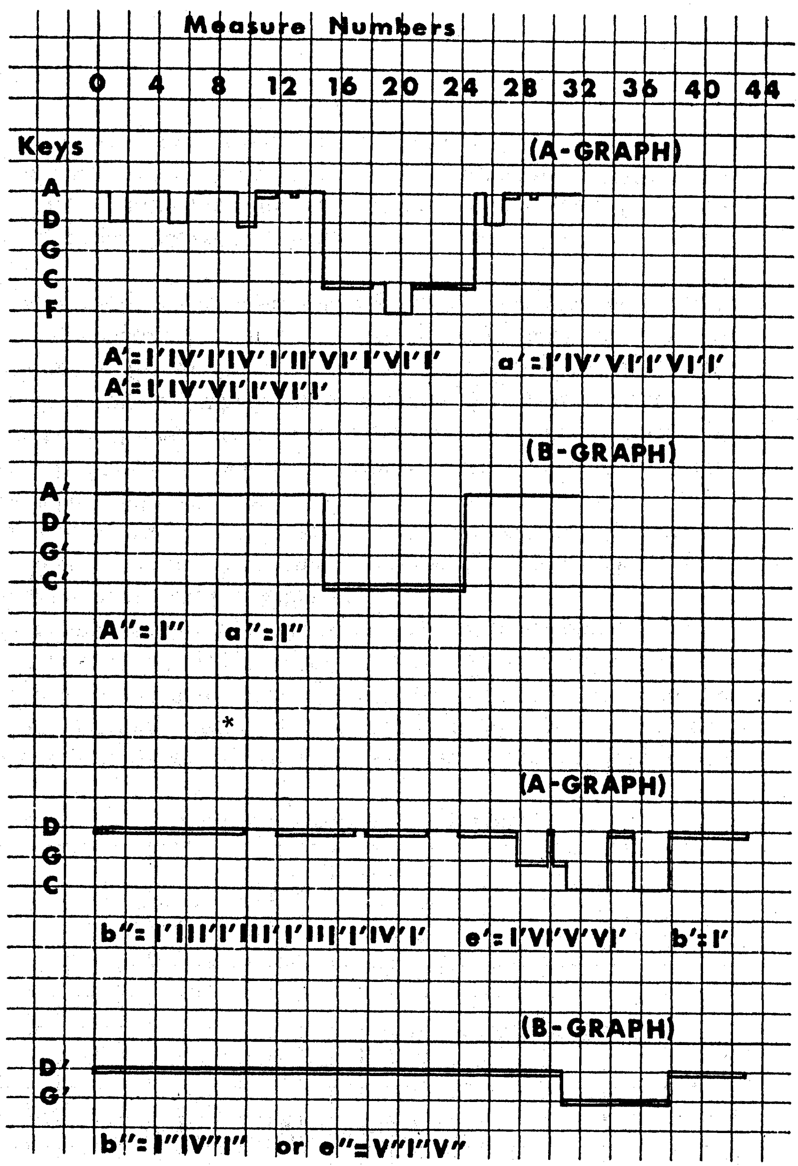


Figure 16. Top: *Die Nebensonnen* by Schubert; Bottom: *Irrlicht* by Schubert

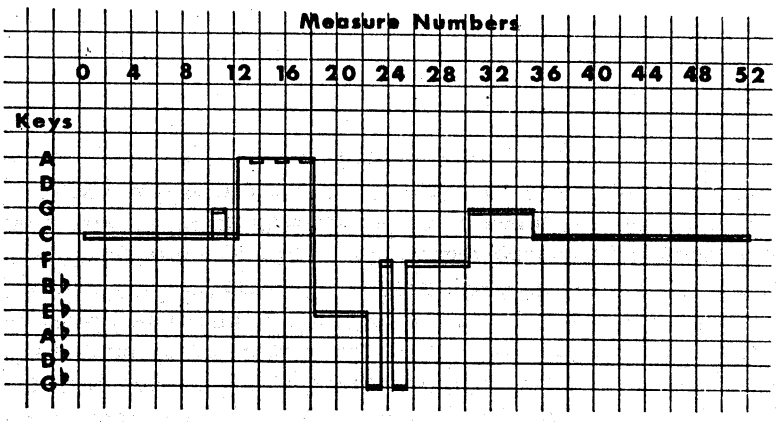


Figure 17. *Das Verlassene Mägdelein* by Hugo Wolf

major key. Therefore, it has a clarifying function with respect to the key, despite its ambiguous nature in isolation. This may serve to explain the importance of the VII ^{6}-I cadence, which preceded the V-I cadence, in fourteenth and early fifteenth century polyphony. It may also explain the contemporary scorn for the diminished seventh chord, for it is valuable only in a diatonic context.

While the seventh scale-step exists as a function of a diatonic major set of tones, VII does not exist as a chordal function of a major key, for the diminished triad is rootless. Due to the univalence of the tritone, however, VII can be considered an incomplete V, as can the diminished seventh chord. Theorists have often considered them thus. In a minor key the diminished triad is more ambiguous than it is in a major key, for it exists on two different scale steps. The diminished triad on the seventh scale step of the minor key is an incomplete dominant, as it is in the major key. Even the diminished triad on the second scale step of a minor key has some dominant character, due to its being a part of the single-rooted dominant ninth chord.

In a major higher-order key there is no VII′ function: as in a minor higher-order key there is no II′ function, due to the fact that the triads on the analogous scale steps are diminished.

One source of the error in contemporary music theory which uses Roman numerals, yet permits the key to consist of the twelve-

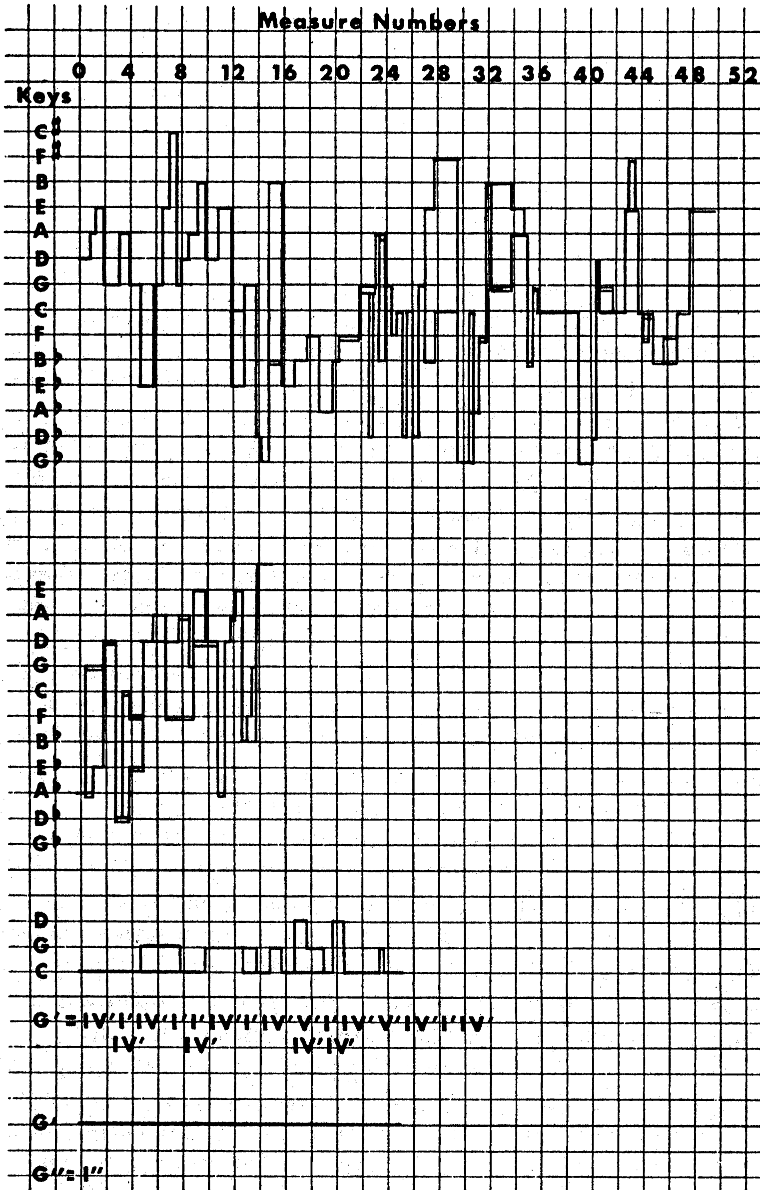


Figure 18. Top: *First Piano Sonata,* First Movement by Hindemith; Middle: *Sechs Kleine Klavierstucke, V* by Schoenberg; Bottom: *Valse* by Stravinsky

tone rootless set, lies in the failure to distinguish between the different orders of single-rooted sets: (1) the chord, (2) the diatonic set or key, (3) the higher-order key, which is composed of keys, or of other higher-order keys. Because all of these entities have a common property, namely, the possession of a single perceptible common measure or predominant root, they were not differentiated from each other. Furthermore, as composers in the late nineteenth and twentieth centuries gradually substituted the twelve tone rootless set for groups of seven tone diatonic sets, the theorists abandoned their clear definitions of key as seven particular tones and allowed the experience of hearing a tonal center to be the defining property of key. Thus a subjective, private definition of the central concept of music theory supplanted the objective, public one. The idea of key was distorted to fit the practice of composers, since people did not want to admit that the practice no longer fit the idea.

The differentiation between the Roman numeral which stands for a chord in a key, and the Roman numeral which stands for a key in a higher-order key, allows us to make the following observations regarding progressions of chords and keys:

1. The V′ which indicates the dominant key of a minor second-order key is a minor key, while the V which indicates a triad in a minor key is major.

2. Many kinds of third relation can now be differentiated according to the relationship between the keys in which the chordal third relation takes place, as well as the relationships between the chords themselves. For example, in the second movement of the Schubert sonata in B-flat, Opus Posthumous, D. 920, third relation occurs between a G-sharp major triad and a C major triad. At the same point in time a key change from C-sharp minor to C major occurs. This can be regarded as a form of change of second-order mode (and key!) from major to the parallel minor (which occurs so frequently in the music of Schubert), for the key of C-sharp minor is VI′ in the second-order key of E major, while the key of C major is VI′ in the second-order key of E minor. Thus we can distinguish between third relation which takes place as a result of a change from the major to the minor second-order mode, and third relation which does not; as for example, the third relation between the first two chords (C major to A major) of the last movement of the Schumann Fantasy in C.

3. Progression from V to IV between chords of a key is rare in the Baroque and classical and Romantic periods, while progression from V′ to IV′ between keys is frequent in the same periods.

4. Varying opinions as to the importance of the III function can now be resolved. For within a key the III triad is of little importance due to the remoteness of its root from the root or tonic of the key. But III′, as applied to a key which is a member of a higher-order key, is extremely important if that higher-order key is minor, for III′ of a minor higher-order key is the tonic of the relative major higher-order key. A frequent key progression within a minor second-order key in Bach's music is I′, III′, V′, which substantiates Schenker's theory of the unfolding triad.

5. A temporal overlap between keys may take place, resulting in a transitional bitonal area. This is similar to what happens melodically when a suspension occurs. It is also similar to what happens harmonically when a chordal suspension occurs, or when V and I occur simultaneously, as they often do even in Bach. The most frequent such temporal overlap with respect to keys is that which occurs in the augmented sixth chords. These chords are bitonal, for they contain the leading tone of the dominant key, in addition to tones of the tonic key, among which is the sixth scale step, which is not a member of the dominant key. Key overlaps occur especially frequently in Brahms, along with metrical and chordal overlaps.

Just as we can speak of non-chord tones, or non-harmonic tones, so also can we speak of non-key tones. They can arise only in music which uses both single-rooted chords and keys, for if they occur in music which uses only keys (for example, Gregorian chant) they change the key. If, in a passage where the key has been established by the presence of the set and the principal chords of the set, we find either the tonic or the dominant chord in arpeggiated form with a leading-tone given to each tone of the chord, then these leading-tones are non-key tones. They act as ornaments to the chord tones. If they occur as ornaments to any other chord than the tonic or dominant, they may shift the emphasis to the key of that chord, and thus we can no longer call them non-key tones.

Summary

1. The paradox of Roman numeral analysis can be eliminated by differentiating between keys which are sets of tones, and higher order keys which are sets of keys.

2. "Function" is defined as a particular position in a single-rooted diatonic set.

3. Roman numeral analysis, in which the Roman numeral is a name for a function leads to increased knowledge of the music being analysed. Nominal Roman numeral analysis in which the Roman numeral is only a label for a set of tones is of no value, except as a terminological exercise, for it can be applied indiscriminately to any music whatever by means of enharmonic equivalence.

4. Some music of the late nineteenth and twentieth centuries is qualitatively different from all preceding western music with respect to the use and compounding of single-rooted sets of tones.

5. The tritone is important, not because of its properties out of context, but because there is only one in any major diatonic set.

6. The key as a function of a higher-order key, is treated differently from a chord as a function of a key.

7. Temporal overlap between keys can exist, as, for example, in the augmented sixth chords.

8. One of the most important style determining properties which has frequently been overlooked by theorists is the way in which diatonic sets, if present, are combined.

Chapter Nine

Modulation

IT IS now possible to define modulation (as it has often been defined in the past) as a change of the basic diatonic set. Such modulations are shown on the graphs by the change in the horizontal level of the line of the A-graph of each composition. (See pages 58-63.) A modulation also takes place on the A-graph when the line changes from single to double, thus indicating a change from major to the relative minor sets, or vice versa. This is the closest possible modulation, the change of a set to its relative set, as the language shows. For, while it is a change of set, it is also reminiscent of the change of mode within a single set, from Ionian to Aeolian.

Higher order analogous "modulations" which are similar to, but not identical with, those between changes of the basic set can also be distinguished. These are the changes between the higher-order sets which are demonstrated by the B-graph of each composition.

It is even possible to speak of an analogue of modulation which occurs when chords change, or even when the single tone changes. In these latter two cases we need no word for it, but in the case of the key change we do need the word. For the changes in the higher-order keys depend for their existence upon the prior existence of the changes between keys. Therefore, whatever we choose to call the change between higher-order keys, the word "modulation" should be reserved for the changes between basic diatonic single-rooted sets.

It would be a semantical absurdity to take the name away from a basic idea and apply it instead to the derivative ideas, which depend for their very existence upon the prior existence of the basic idea.

If this definition of "modulation" is accepted, then we can proceed to distinguish between diatonic and chromatic modulations.

Diatonic modulation is that which takes place between two keys which are functionally related to each other as are the triads of a single key. All other modulations are chromatic. According to this definition both the Bach examples are diatonic. Similar graphs of most late romantic and contemporary compositions show them to be highly chromatic.

This definition of modulation makes it possible to attempt a connection between the semantics, or emotional meaning, of music and the syntax, or pitch relationships, of music. For the same graph which displays the syntactical factor of the relationships between sets can also be used to demonstrate the emotional qualities of the music. I conjecture that, other things being equal, which they never are, a change of set up a fifth is positive (happy, bright, hopeful) and a change of set down a fifth is negative (sad, dark, despairing). The adjective may vary with the individual listener, but the direction will remain the same for all listeners.

Up to a certain point, the greater the number of fifths between sets the greater the emotional effect, however, beyond that point the law of diminishing returns sets in, and only increasing ambiguity results.

Similarly, up to a certain point the greater the ambiguity the more negative (sadder, darker) and the less the ambiguity the more positive, (brighter, happier) the music will be. This statement may apply not only to relationships between consecutive roots of diatonic sets, but also to relationships between roots of chords and relationships between single tones of a melody.

Thus a possible explanation for the emotional effect of the change of set from major to the parallel minor may be that the cardinal center drops three fifths while the ordinal center remains the same. Furthermore, an explanation of the emotional effect of the change from minor to the relative major may be that the ordinal center and the cardinal center become the same, thus diminishing the degree of ambiguity.

The validity of these conjectures can easily be tested empirically by the interested listener by watching the graph of the piece while listening to it.

Summary

1. Modulation is the change from one single-rooted diatonic set to another. Higher order modulations between higher order sets may also exist in music.

2. Chromatic modulation is that which takes place between two keys which are not functionally related to each other as are the triads in a key.

3. Different modulations have different emotional effects.

Chapter Ten

Pitch And Meter

METER, LIKE pitch, is number mixed with time. That is, meter is repetition, or repetition of repetitions (etc.), the smallest element of which is one particular interval of time, marked off, or limited, by two points in time. The smallest possible metrical repetition consists of two of these time intervals, marked off by three beats, or points in time. All meter, like all pitches, can be expressed in terms of numbers.

Pitch and meter appear to be different phenomena to us because of the different way in which we perceive them; the faster periodicities of pitch are perceived as tones, while the slower periodicities are perceived as meters.

There are also some objective differences between pitch and meter; material periodicities are discrete, while pitch periodicities are formed by continuous pressure changes. Metrical "tones", that is, frequencies per second, have no overtones; they are not compounds of partial frequencies as are pitches. However, they have the analogue of undertones due to the process of factorization in human perception of meters. We tend, for instance, to hear four beats as two groups of two beats. In perception of meters, we are consciously aware of the individual pulses which make up the metrical tone, while this is not the case in the perception of pitches, which we hear consciously as continuous phenomena.

Two distinct problems arise in the discussion of meter as they did in the discussion of pitch: the problem of determining what the meter is in a given composition, and the problem of the properties of the metrical relationships once we have decided what they are.

Meter is made manifest in music by the patterns of accents. There are many factors which contribute to the determination of the position of the relative accents in music, among which is the notation of the music: the position of the bar-lines with respect to the notes. This is of course the most superficial of all the factors. It is possible, however, to notate music incorrectly; it is in fact a widespread practice as a means of attaining the illusion of complexity by means of obscurity. It can be found as early as Schumann and Brahms, as well as increasingly often in later works. There are some compositions which have had to be entirely rescored to facilitate performance. The way to notate meter incorrectly is to place the bar-lines in such a way that they are unrelated to the periodic accents, or to change the notation of the meter constantly, while the beats and groups of beats continue undisturbed under the fancy notation.

Today the connection between bar-line and accent in music is strongly disputed, because of the fact that much contemporary music, like prose, has no regular groupings of beats, although the bar-line remains as a necessity for the performance of the music in concert.

Although many people assert that the only function of the bar-line is to keep the players together, still there exists a great mass of music in which the periodicities represented by the bar-lines are very real, and an essential part of the music. Harmonic change over the bar-line is an important element which contributes to this larger metrical periodicity. Treatment of dissonance is another, for if dissonance is more carefully treated on the first beat after the bar-line than on any of the other beats, this will tend to set up a metrical pattern.

The establishment of meter is a cumulative process, for while pitch relationships help to establish it, so also does an established meter itself influence, by selection, the pitch relationships themselves, by means of the emphasis it places upon tones.

There are other, more obvious forms of accentuation in music, some of which are duration of a tone, voluntary accentuation by touch or loudness, position of a tone at the peak of a melody, repetition of melodic patterns, etc.

One important element which is usually ignored in discussions of meter, comes from the performer or the perceiver. I shall therefore call it subjective meter, not, however, implying by this term that it is arbitrary, illusory, or unreal, only that it has its source in the person rather than on the printed page.

Subjective meter results from a built-in tendency of most people to accept periodicity for the sake of economy in order to eliminate unnecessary effort. Walking, for instance is normally periodic, how wasteful of energy it would be if it were not! And how difficult to get anywhere! We have a natural tendency toward consistency which could even be called habit.

This tendency to make things regular exists in varying degrees in various people; one can consciously resist it if one wants to, but it is impossible entirely to eliminate it, for it lies at the center of life, in the heart-beat. Even the voices of silence speak in periodicities if they speak to a living person.

Today much of our theory about meter, as it is about pitch, is derived from contemporary music and poetry, which often has no objective metrical periodicities. We mistake the violation of the norm for the norm itself, and finally come to the conclusion that meter, if it exists at all is entirely subjective, and that the only function of the bar-line in music is to enable the performers to keep together.

It is the privilege of the artist in his art to break any rules he wishes, to violate any norm as he sees fit. He must have absolute freedom to create, just as his audience should have absolute freedom to praise or to castigate. The artist's intuition may be far ahead of the theorist's intellect. But the freedom of the artist does not extend to the theorist. For anyone, either artist or theorist, to deny the existence of a norm in other art than his own, to obliterate the idea, to remove a meaningful word from the language, is to impoverish the world.

The properties of metrical relationships will now be investigated apart from the difficulty of their establishment. There is no doubt that they do exist in some music; this is sufficient reason to discuss them.

Since meters, like pitches, can be represented by equally spaced points in time we see that much of what has been said about pitch applies directly to meter. The following pictures represent metrical units equally as well as they represent pitch units:

meter:	duple	triple	2 against 3
pitch:	octave	twelfth	fifth

Meter	Picture	Numerical Representation	Consonance I	Consonance II	No. of Roots
2/4	... . .	1:2	1	2	1
3/4	.... . .	1:3	1	3	1
4/4	....	1:2:4	1	4	1
5/4	....	1:5	1	5	1
7/4	....	1:7	1	7	1
6/8	....	1:3:6	1	6	1
9/8	....	1:3:9	1	9	2
12/8	....	1:2:4:12	1	12	1

Figure 19. Pictorial Representation of Meters as Points in Time

Figure 19 gives the pictorial representation of all the familiar meters. It also gives the pitches represented by the same patterns, as well as their roots and consonance values.

We can speak of the metrical analogue of consonance, root, chord, non-harmonic tone (the temporary syncope is the counterpart of the non-harmonic tone), and function. We shall see that similar principles apply to chord construction and metrical construction.

The metrical "tone" is a given metrical periodicity. Thus a metrical "chord" is a simultaneous set of periodicities, each of which corresponds to a single metrical "tone". Metrical chords can be expressed by sets of integers. The integral representations of the metrical chords are given in Figure 19. These representations differ from the meter signature, for in the latter the unit is represented by ¼ rather than by 1.

Consider a four-measure phrase in 4/4 meter, each beat of which is subdivided into four sixteenth-notes. Figure 21 shows the metrical chord, whose numerical representation is 1:2:4:8:16:32:64.

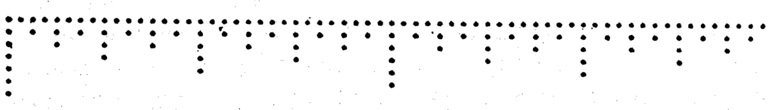


Figure 20. Pictorial Representation of the Metrical Organization of the Four-Measure Phrase

There are seven tones in this chord, all of which are separated by the interval of an octave. Thus the metrical sonority is six octaves, and it possesses the extremely strong reinforcement of the fundamental which such a sonority possesses.

If the phrase is in triple meter its numerical representation is 1:2:4:12:24:48:96. Its sonority is that of an open twelfth, with octaves both above and below the twelfth. If it contains 3 against 2 at any metrical level, it contains the interval of a fifth.

Any further duple compounding of metrical units has the result of adding lower octave reinforcement. Any further duple division of metrical units has the result of adding higher octave reinforcement.

The great mass of music, from the fifteenth to the late nineteenth centuries, the meter of which is compounded of duple and triple units, achieves a metrical sonority which is no more dissonant than the principal sonority of the middle ages: the open triad, that is, the interval of a fifth, with or without octave reinforcement.

The numerical consonance measures defined for sets of tones, apply equally to sets of metrical tones, as does the numerical definition of root. The absence of any repeated metrical units is the metrical equivalent of noise, as opposed to musical tone. Our tolerance of metrical dissonance is far less than it is of pitch dissonance, as is shown by the great prevalence of duple and triple metrical units over quintuple or septuple ones.

In metrical chords, as in pitch chords, the root is the lowest power of two which is present. There is a difference, however, for no metrical chord can exist without the presence of 2^0, or 1, for the presence of the meter itself implies a periodicity of 1. There is another important difference between metrical and pitch chords which results from the different degree of accuracy with which we perceive pitch and meter. This difference concerns the octave. We can speak of octave doubling in pitch relationships, for the two tones are so closely related as to appear to be almost indistinguishable, and because the set of partials of the upper tone is completely contained in the set of partials of the lower tone. But there is no possibility of our failing to distinguish between metrical octaves. Furthermore, there are no metrical overtones, but only undertones. Due to the fact that meter takes place in the range of our conscious perception of unit time intervals, while pitch does not, differences between metrical tones are far more striking than are differences between pitch tones. We would never mistake a half note for a whole note. Thus, metrically speaking, an octave confers a root, while tonally speaking it does not necessarily do so.

From Figure 19 we see that of the meters given, only septuple and quintuple meters have no roots other than 2^0. We also see that their metrical sonority is minimal. These facts no doubt account for the rarity of occurrence of these meters in music before the twentieth century.

We can speak also of metrical function, analogous to pitch function. A given metrical tone has a particular function relative to the root of the single-rooted metrical set to which it belongs. Since we can distinguish not only between the metrical tones, but also between single pulses of those tones, we shall ascribe to each particular position, or pulse, in a metrical structure a particular function according to the quantity of metrical tones, or functions, to which it belongs. For example, in the four-measure phrase of 4/4 meter of Figure 20, the metrical function of the first beat of the first measure is 7, for it belongs to seven different metrical tones or functions of the single-rooted metrical set. Suppose that the first beat is divided into sixteenth-notes. Then the metrical function of the second sixteenth-note is 1, for it occurs in only one of the metrical tones. That of the third sixteenth-note is 2, for it occurs in two of the metrical tones. Thus to each position of this metrical hierarchy there corresponds a metrical function (which is a function of functions) which contributes to the total function of the tone which appears in that position.

A possible principle for the construction of good pitch or metrical chords is the following: skip no prime partials. That is, make sure that a power of each prime partial smaller than the highest one which appears is present. Quintuple meter (1:5) is the only occurrence of a metrical violation of this principle in the literature until the twentieth century, and its occurrence is rare. The major third, 4:5 occurs, it is true, as a perfectly good pitch interval in violation of this principle, but that may be due to the fact that pitches have overtones, and the missing prime partials are supplied by them, while metrical tones have no overtones to supply the missing partials.

There may be in music an overlapping area of metrical periodicities and pitch periodicities, if the smallest metrical periodicities are fast as twenty beats per second, which is the approximate periodicity of the lowest pitch on the keyboard. The fastest marking on a metronome is 208 beats per minute, which is 208/60, or 3.46 beats per second. That unit can easily be divided mentally into two metric units, and the writer can with some difficulty divide it into four metrical units mentally. Suppose that 4 times 3.5, or 14, were the metrical perception threshold. Twenty cycles per second is a rough estimate of the pitch perception threshold. In the area of between 14 and 20 beats per second we perceive the periodicities as a vague rumble, neither pitch nor meter.

Can we conjecture from this fact that there may exist some principle by which the tempo of a composition is ideally determined by the pitches which compose it? It seems to the writer to be the case, for if we choose to maintain a continuity of successive division of temporal units, then within a certain range the pitches themselves would determine the tempo. This substantiates the intuition of musicians who feel a certain tempo to be innately implied by a given composition.

The greater the quantity of intermeshing commensurable periodicities, whether metrical or tonal, the greater the periodic complexity of a composition. Let complexity be defined as follows: The greater the quantity of groups within groups, the higher the degree of complexity. The familiar ballad meter, 4/4 has often been considered the simplest of all meters. I submit that, on the contrary, it is because of its complexity, or high degree of order (negentropy as the information theorists call it) that it is so often used, and thus familiar.

Often "complexity" is used in another sense, to describe the condition when the various factors contributing to the determination of the meter conflict. This is a subjective definition of complexity, which depends upon the ease of discovery of the periodicities rather than upon the relationships between the periodicities. It is the result of confusing the process of analysis, with the music which is being analysed. Often, because it is difficult to discover the underlying meters of a composition, people humbly assume that the meter is complex. Similarly, if the meter is clear and evident, it is regarded as simple because its discovery is so easy. This is like the confusion, in the realm of ideas, of obscurity with profundity.

Real complexity of meter or pitch relationships cannot occur unless a periodicity is clearly established on one level. There can be no groups within groups if there are no groups. The establishment of this built-in reference level is the purpose of roots in music, whether they be tonal or metrical.

Summary

1. Meter is number mixed with time.

2. Meter and pitch are similar in that they are both temporal phenomena which can be spatially represented by equally spaced points on a line.

3. The problem of determining what the meter is in a given piece must be distinguished from the problem of determining the properties of metrical relationships.

4. Pitch relationships influence meter, and an established meter in turn influences pitch relationships.

5. One element of meter is subjective, but real.

6. Our tolerance of metrical dissonance is far less than it is of tonal dissonance.

7. In a single-rooted metrical chord, each metrical position has a function whose value depends upon the quantity of metrical tones to which it belongs.

8. Tonal and metrical complexity are defined as varying directly with the quantity of groups within groups, or levels of hierarchical structure, which are present in a composition.

Chapter Eleven

Tonality

IT IS now possible to suggest a definition for "tonality": Music is tonal if the majority of its adjacent tones, whether simultaneous or consecutive, form single-rooted sets. Tonal music thus possesses one or more perceptible common measures, or roots, among its tones. Some music is tonal and some music is not tonal. Some music is more tonal than other music. The measure of tonality is the following: The greater the quantity of simultaneous common measures, or roots, to which the tones of a composition are related, the higher the degree of tonality. Thus the metrical hierarchies, as well as the tonal hierarchies, contribute to the tonality of a composition.

A theoretical distinction has been drawn here. There exists music which is minimally tonal according to this definition, but which may not be heard as such, due to the too frequent change of root or to the absence of two or more simultaneous common measures. Whether such music will be heard as being tonal may depend to some extent on the listener. But the increase in tonality as described above will be heard by all listeners, despite the fact that the border line may vary from listener to listener.

The greater the quantity of different pitches of the composition which possess a single common measure, the more tonal is the music. It is only by means of hierarchical ordering of the pitches that this ratio can be increased from 7:1 to 12:1, since the seven-tone set is single-rooted, and the twelve-tone set is not. Music which uses keys as well as single-rooted chords will be more tonal than music which uses single-rooted chords only, for it is by means of the key, and combinations of keys, that the higher-order single-rooted set which includes all the twelve tones can be constructed.

Bach's music is tonal; Schoenberg's atonal music is atonal. They are not similar; on the contrary, they are qualitatively different.

Just as there are varieties of tonal music, so, on the other side of the border, there are varieties of atonal music. There is atonal music which has no common measure conceptually, and atonal music which has no common measure perceptually. Consider music which uses the tones of the twelve-tone equal-tempered set, but which does not group the tones into either single-rooted chords or diatonic keys. The intervals of such music will be principally seconds and sevenths. Since these are the intervals for which no clear paradigm intervals exist conceptually (see Chapter I), the ear has no prototype to which to correct them, and they remain in their pristine irrational state. Altogether apart from any considera-tion of human perception thresholds, this music has no conceptual common measure. The irrational is that which has no common measure.

The unifying factor in such music, if it exists, must be an arbi-trary compound of tones, for example, the tone-row. Such music is ordered in a qualitatively different fashion than are the partials of a single tone. It is thus a formalism whose structure is inconsistent with the nature of its elements.

If the music were constructed of the tones of the Pythagorean twelve-tone scale instead of the equal-tempered twelve-tone scale, then, it is true that theoretically it would have a common measure, for the Pythagorean scale can be represented by a set of natural numbers, of which 1 is the common measure. But if each tone of the Pythagorean twelve-tone scale is expressed as an integer, then the smallest of these integers is 2^{17}, or 131072, an upper partial so remote that surely no one would maintain that it could be heard as related to 1, its fundamental. For the root-strength dimin-ishes as the power of 2 increases, whether or not the scale of di-minution suggested on page 32, Chapter III, is accepted. The perceptibly rootless augmented fourth has a root strength of 2^5. 2^{17} is a very much greater number than 2^5.

If the music were constructed from the tones of the just twelve-tone scale derived on page 46, Chapter V, which is the least arbi-trary just twelve-tone set in existence, then, if each tone of this set is expressed as an integer, the smallest integer is 2^{10}, or 1024. The same argument which applies to the Pythagorean twelve-tone scale applies to this scale.

If the tones of a Pythagorean scale are expressed as integers and thus equated to partials of an overtone series, there is a gap of seventeen octaves between the tone which is closest to the funda-mental, or common measure, and that fundamental. In the case of the just twelve-tone scale mentioned above, the gap is ten octaves. These gaps render the common measure imperceptible. The only way to close these gaps, these discontinuities, and still be able to include all the twelve tones, is by hierarchical grouping of the tones into single-rooted subsets whose roots in turn can be grouped into single-rooted subsets, etc. (Might these single-rooted subsets, these gestalts, be analogous to the discrete quanta, which, in the present physical view, constitute matter?)

According to the definition of "tonality" suggested herein, all Western music until that of the late nineteenth and twentieth centuries is tonal. For the single-rooted diatonic set, or the single-rooted pentatonic set is the basis of all Western music (and of most Eastern music too, for that matter). What has often been called modal music is a species of tonal music, for the modes are variants of the diatonic set.

In the period of what is usually called tonal music single-rooted chords, single-rooted keys, and metrical hierarchies were present simultaneously. But even before this high development came into existence, the hierarchy of the diatonic set was ubiquitous. It was assumed without question; so deeply embedded was it in the unconscious that it never rose to consciousness, and thus it could be gradually abandoned without people's realizing what was happening. From the middle of the nineteenth century the metrical and tonal hierarchies (two aspects of the same thing) which had previously reinforced each other began to be set against each other. We see this in Brahms' counter-rhythms and melodic, chordal, and key suspensions and overlaps, in the altered harmonies of the Romantics, the roots of which harmonies could be grouped into keys, but the tones of which increasingly often departed from the keys. The four-measure phrase, together with the metrical units represented by the bar-lines, gradually disappeared, and the chord took precedence over the key. The very ideas of bar-line and key began to get lost. Witness the assertions of the tyranny of major and minor, and the tyranny of the bar-line. The bass took precedence over the root, and, similarly, the idea of root itself got lost, and only the ideas of sonority and consonance remained. Consonance then, when it was not entirely cast out in favor of dissonance, together with sonority variation, became the basis of much harmonic technique. Some composers kept it constant; some varied it in an orderly continuous fashion.

Neither of these practices results by itself in tonality, as defined herein. Something more is necessary, and that something more is the root, or common measure.

Summary

1. Music is tonal if, and only if, the majority of its adjacent tones, whether simultaneous or consecutive, form single-rooted sets.
2. The greater the quantity of simultaneous common measures to which the tones of a composition are related, the more tonal is the composition.
3. Only by means of hierarchical structure can the ratio of different pitches to a single common measure be increased from 7:1 to 12:1.
4. Equal-tempered, dissonant, non-diatonic music is irrational.
5. All Western music which preceded the impressionistic development in France was tonal. A large portion of twentieth century music is atonal.

Chapter Twelve

Aesthetics

IT IS the purpose of this final chapter to demonstrate that, if we accept certain interrelated aesthetic axioms, then tonal music is absolutely better, relative to those axioms, than atonal music. The axioms are the following:

1. Unity is good.
2. Variety is good.
3. Order is good.
4. Consistency is good.
5. Expressive symbolism is good.
6. Continuity is good.

I will attempt to show that tonal music has potentially more of all of these properties than has atonal music.

Greater unity is possible in a tonal composition than in an atonal one, for all the means of achieving unity available to atonal music are also available to tonal music, while there is one unifying factor in tonal music which is not available to atonal music.

This is the perceptible common measure.

Why is it good to have a common measure? The New York housewives did not ask that question when it was discovered that the city butchers had altered their scales to obliterate the common measure, for the answer was too obvious. The answer is also implicit in that rich source of insight, the language, for the absence of a common measure is the literal, precise, mathematical meaning of "irrational".

Contemporary physicists and philosophers have pointed out that we cannot be certain that our measuring rods do not change as their spatial and temporal coordinates change. But no one suggests that for that reason we should change our measuring rods during the course of an experiment. If we did we could no longer contact the objective world. "Music is counting performed by the mind without knowing it is counting,"[1] says Leibnitz. Perception of light, as well as of sound, is a form of unconscious counting of the impinging periodicities. Counting is only possible by means of a common measure.

The rejection of a common measure of equally spaced points in time, each of which differentiates the past from the future by asserting the present, is symbolic of the rejection of a common measure of objective time, which can be shared by more than one person, thus rendering its passing more inescapable. Periodicity stands for circumstances over which we have no control, like those boundary conditions of life: the passing of time, the existence of an uncaring world outside of us, and the certainty of death. To reject periodicity gives the illusion of freedom from the outer world by making it appear to be illusory.

The concept of unity which is achieved by means of the common measure is similar to Heinrich Schenker's idea of tonal unfolding.[2] The tone which is unfolded is the tone whose frequency is the greatest common measure of the music which unfolds it. This tone is unfolded into other tones, which are again unfolded into other tones, and so forth. The only way to unfold a tone so that it will be heard as such is to unfold it into single-rooted sets which have the tone as root. Tonal music in which the hierarchy is developed to the extent of the presence of chords, keys, and higher-order keys does this. The single tone, or common measure, which is unfolded, is the unity which is present in many compositions, all of which appear to be entirely different from one another. It is the invariant, the essence, the common property, the quidditas, of which the composition itself is the variant, the existence, the diversity, the haecceitas.

Greater diversity is possible in tonal music than in atonal music. The bargain which the composers of the late nineteenth and twentieth centuries made when they exchanged the single-rootedness of the seven-tone set for the five additional chromatic tones was possibly the worst bargain in history, if its aim was the increase of available tonal material. For tones come into full existence only in relation to other tones. The addition of the five chromatic tones to the seven diatonic tones eliminates the functions of the tones, and

[1] Leibnitz, in a letter to Goldbach. Leibnitii Epistolae, ep. 154.

[2] Schenker, Heinrich, *Harmony*, Chicago University Press, Chicago, 1954.

it is by virtue of the multiplicity of its functions that a particular tone assumes a character. In a diatonic system of keys, chords, higher-order keys, and metrical compounds, a single tone can take on a very large, possibly infinite, set of identities, according to the particular combination of functions it possesses, while in a rootless, non-metric system the single tone is always and only a frequency, barely, if at all, distinguishable in character from its neighbouring tones. For a tone acquires its function from its context. In dissonant rootless music, the heard context itself is obliterated, for each new tone destroys the memory of the previous tones. Without a root or tonic as a reference tone, the mind cannot (consciously or unconsciously) relate all the incoming tones to each other, so each tone stands alone and acquires no function.

A tone takes on its identity according to the set of functional identities it possesses. A tone takes on its function according to its hierarchical position relative to the roots of the single-rooted sets of which it is a member. It also takes on its identity according to its metrical function, which, as we have seen in Chapter X, depends upon the presence of single-rooted metrical chords. The functional identity of a tone, therefore, is determined by the metrical organization of the composition as well as by its pitch organization.

To give an example, I shall calculate the possible functional identities available to a single tone in the Agnus Dei of Bach's *Mass in B Minor;* the graph is shown on page 60 in Chapter VIII. In this composition the four-measure phrase is established, as is the single-rooted chord, the diatonic key, the second-order key, and the third-order key.

A single tone assumes its tonal function by virtue of its relative frequency, and its metrical function by virtue of its position as a pulse in one or more particular metrical tones. The possible pitch functions multiplied by the possible metrical functions (since in this case the two are independent) gives the total possible functions of a single tone. The total possibilities are 3 (functions of a single-rooted chord) times 7 (functions of a key) times 6 (functions of a second-order key) times 6, (functions of a third-order key). Since the four-measure phrase of 4/4 meter is established, whose fastest tone is the sixteenth-note, we have a metrical single-rooted chord of seven tones each separated by the interval of an octave. Thus, in this composition, a tone can possess one of seven metrical functions according to which of the metrical octaves it belongs. In order to find the total possible combinations of functions which a single tone can assume according to context, we multiply all these possibilities together: 3 times 7 times 6 times 6 times 7, which equals 5292. (The actual number is somewhat less than this, for the functions are not entirely mutually independent, but it is of a similar order of magnitude.)

There is no limit to the quantity of possible tonal and metrical compounds due to the fact that, by means of enharmonic equivalence, the twelve-tone equal-tempered set is the set of all possible diatonic sets.

In atonal music with fluctuating meter a given pitch takes on no functions.

Thus, the addition of five tones to the seven-tone set, as well as the dissolution of the metrical compounds, actually achieved a great impoverishment of the composer's material rather than the enrichment which it was intended to achieve. By this Faustian bargain the composer gained absolute freedom of choice, but was left with little to choose from.

A greater degree of order is possible in a tonal composition than in an atonal one. A new semantical difficulty enters if we wish to compare tonal and atonal music with respect to the order attainable in each, for "order", like "tonality" is a word which has been misused and redefined to such an extent that there are people who divest it of all meaning other than an arbitrary serial occurrence in time. The similar situation of these two words in our culture is not surprising, for their meaning is very similar, tonality being the tonal manifestation of the abstract idea of order.

By "order" is meant here hierarchical structure, architectonics, the essence of which is groups combined within groups, which are again combined within groups according to a similar principle, and so on. An ordered structure in this sense is the exact opposite of a random structure, or the "heat death" as the physicists call it.

Tonal music can be more highly ordered than can atonal music, for however high the degree of order an atonal composition may have, a tonal one can have more. For the potential order of a tonal composition extends down into the microscopic level of the structure of the single tone itself, while this is not true of any hierarchical order which may be present in atonal music. A measure of the degree of order is the quantity of groups within groups which occur in the set under consideration. However high this number may be in atonal music, it can always be higher in tonal music, for in tonal music the group of partials present in a single tone is a part of the order of the whole, while in atonal music it is not.

A tonal composition can be more consistent than can an atonal composition. The principle of suggesting a fundamental by the use of its lower partials is the principle behind the construction of single-rooted sets, whether they be chords or keys. Chords are constructed by the use of tones related to each other as are the partials themselves; keys are constructed by the use of the intervals between the lower partials. So also are higher-order keys constructed by the use of these intervals, as well as by analogy with the relationship of the chords within a key to the tonic of the key.

Tonal music, then, is internally consistent. But it is more than this, for it is also consistent with the inner nature of its elements; the single tones. The principle of ordering of the tones by the composer is the same as the ordering of the partials into a single tone. The subjective element, or that which is contributed by the composer, follows the same principle as the objective element, the structure of the tone.

The electronic composers, in order to do away with this inconsistency, have considered the alteration of the single tone by stripping it of its partials. This can be done electronically. But nothing, short of an alteration of the human physiology, can do away with human experience of the overtone series, for it exists in the tones of the human voice, as well as in the tones produced by musical instruments.

Should the ultimate attempt in the avoidance of repetition be made; that is, the alteration of the periodic nature of the single pure sine-wave, then the tone itself would vanish, for a tone is distinguishable from noise only by its being a periodic phenomenon. If "music" were to be composed whose basic element were noise, the argument for consistency in this chapter would no longer be valid. But would such "music" any longer be music?

In a recent article in the Journal of Music Theory, a new concept of tonality was proposed. Tonal music was defined as music which

. . . unfolds through time a particular tone, interval, or chord.[8]

[8] Travis, Roy, *Toward a New Concept of Tonality,* Journal of Music Theory, Yale School of Music, Vol. III, No 2., November 1959, p. 261.

If the set were single-rooted, then the definition would agree with the one suggested in Chapter XI, and the music described by it would stem from the single tone. But if the set were not single-rooted, then the idea is meaningful, but the music it describes would not be as consistent as tonal music as defined herein, for it would not be consistent with the nature of its elements, the single tones.

Music whose hierarchical ordering extends to meter as well as to pitch has an even deeper consistency than tonal non-metrical music, for its consistency extends from the microcosm to the macrocosm. The same principle of set formation appears from the bird's-eye view as from the worm's-eye view. Until recently the physical world was supposed to have this property, for the electron circling around its nucleus was the analogue of the planet circling around its sun. Is there possibly an analogy between atonal music and some contemporary physics?

It is possible to speak of ordinal consistency in tonal music as well as cardinal consistency. Ordinal consistency concerns consistency of direction between tones occurring consecutively in time. Single tones as well as roots and tonics can move up or down by fifths. Thus we can have parallel or contrary motion between these units of differing generality.

In tonal music all the possibilities of expressive symbolism can occur which can occur in atonal music; in addition, there are some possibilities available to tonal music which are not available to atonal music. The expressive symbolism of change of mode and key, either harmonically or melodically or both, is not possible in music which has no modes or keys. The expressive symbolism of the non-harmonic tone is not possible in atonal music.

In tonal music the expressive symbolism of direction up or down between tones, roots of chords, roots of keys, or roots of higher order keys is available to the composer, while the only direction possible in atonal music is that between the tones themselves. Since there are more ways to establish direction in tonal music, there are also more ways to establish conflict of direction, whether it be between two melodic lines, between roots and melodic lines, between root progression and key progression, etc. In atonal music the conflict can be established only between tones (dissonance) or between melodic lines. All dissonance used in atonal music is available to tonal music, where its symbolic effect is made even stronger by contrast with consonant passages. But if dissonance is the norm, its expressive possibilities are diminished, for conflict itself has only a relative merit; it is expressive only in relation to the possibility of absence of conflict.

In tonal music the expressive symbolism of ambiguity and clarity is possible, not only between tones, but between roots and tonics as well. Furthermore, the symbolic ambiguity of an atonal passage imbedded in tonality is possible.

Another rich source of expressive symbolism in tonal music lies in the relation of a theme or motive to its harmonic environment. A theme may occur in tonal music in one position relative to the key of the music, and later the same theme may occur again in a different position relative to the key in which it is appearing. An example of this is to be found in the first movement of Schubert's B-flat Sonata, Opus Posthumous, D. 920, where the theme occurs first on the tonic of the key of B-flat major, and later on the third scale-step of the key of G-flat major.

With the gradual downfall of tonality we observe a great increase of extraneous means of expression, that is, expression which arises from some other source than the relationships between the tones themselves. The importance of the virtuoso performer increased to the point where concerts were, and are, advertised, not by composer, but by performer. Articulation and phrasing, special instrumental effects, even to the extent of out-and-out sound effects (as in *musique concrète*), have become increasingly important, possibly in order to fill the expressive vacuum left by the disappearance of tonality.

It is a prevalent idea today that all symbolic expression in music stems from the violation of a norm.

> . . . the frustration of expectation has been found to be the basis of the affective and the intellectual response to music.[4]

In Chapter IX a source of expression is described which is not the violation of a norm, but which is the result of the built-in expressive symbolism of a tone which has a functional identity due to its hierarchical context.

But suppose we grant that it is expressive to violate a norm. Undoubtedly it has a certain surprise shock value. Then we can

[4] Meyer Leonard B., *Emotion and Meaning in Music,* University of Chicago Press, Chicago, 1956, p. 43.

distinguish between music which is expressively self-sufficient and music which is not. The former kind, if it violates a norm, violates a norm which is built into the music. For example, in a tonal composition a non-harmonic tone is a violation of a norm at a deep level; an exceptionally distant modulation or an exceptionally dissonant chord, or an unusual harmonic progression, is a violation of norms which are established by the composition itself.

Music which is not expressively self-sufficient violates cultural norms, rather than norms set up by the music itself. It thus depends upon the education of the listener whether or not he will "get" the expression of the music. Music whose sole source of expression is this sort of violation of a cultural norm is music for the elite. Only those who are educated to its opposite (that is, music embodying the norms which it violates), will be able to appreciate it. It is thus contra-educational music. This is not true of tonal music, the appreciation of which is increased, not diminished, by knowledge about it.

Music which depends for its expression upon the violation of cultural norms or expectations is parasitic, and, like all parasites, destroys its host and thus itself. For if all music depended for its value on the violation of the cultural norm, then in time there would no longer be any cultural norm to violate, and thus no violation would be possible.

Greater continuity is possible in tonal music than in atonal music, for all the means of maintaining continuity in atonal music are available to tonal music; but the opposite is not the case, for tonal music can establish continuity by means of the presence of a common measure. The successive pitches of atonal music are not time-binding, they obliterate each other in the memory of the listener, for they are incommensurate with each other, and there is no common reference tone to which the ear can associate them. Each successive tone destroys the memory of the previous one. Atonal music must rely on more obvious forms of continuity: Wagnerian continous flow, or cyclic form, the apotheosis of which is the serial composition, whether it be of the tone-row variety or the later and more highly serialized compositions of the electronic composers. (For all their emphasis on series, they ignore the most important one, the overtone series.)

Tonal music, on the other hand, can risk sectional breaks, and can permit the enormous thematic variety found, for example, in

Mozart, for it has the presence of one or more common measures to hold it together. If it is composed principally of single-rooted chords within keys within higher-order keys, then it is possible momentarily to dissolve one of these common measures and rely on another. Thus the temporary use, in a tonal context, of dissonant rootless chords need not destroy the continuity, for it can be maintained by the common measure presented by the key. Similarly, the temporary use of sets of tones which do not form keys need not destroy the continuity, if the higher-order key is present to provide the common measure.

The failure of the superficial continuity of atonal music is evidenced by the paradox that it is often atonal music which seems lacking in cohesion, and which never seems to get anywhere, while the music of Bach, Mozart and Beethoven, which is full of barlines, gaps, four-measure phrases, repetitions, divisions, and great climactic cadences, travels continously and appears to transcend ordinary time.

We live in an age when the emphasis in every field is on dynamics, Heraclitean flux, rapid change; where the whole, the gestalt, is asserted and its parts are often ignored. In linguistic analysis the sentence, not the word, is regarded as being the basic unit of language. Some schools of philosophy maintain that a word has no meaning out of context. In musical analysis the phrase is often regarded as the basic unit instead of the single tone. Some mathematicians consider the axiom, rather than the word, to be the elementary unit of meaning.

It is true that the tone takes on its meaning, or function, from the whole, and that each added tone of the whole may alter the relationships between all the other tones, and thus the function of each single tone itself. But without the existence of the relatively static reference tone, the other tones take on no function. Without the static point which bifurcates the time continuum there can be no tone at all, and hence no music.

Similarly, without accepting the present moment of the observer's identity, there can be no aesthetic comparison between styles, for the very existence of the formulation of a style is itself dependent upon the prior existence of a judge who judged, independently of stylistic considerations, a particular set of tones to be music. The absence of these static points which bifurcate the time continuum sets up an infinite regress in both music and the theory of music.

The static and the dynamic, harmony and melody, permanence and change, part and whole, are symbiotic ideas, which, like the binary stars, depend upon each other for their very existence. To destroy one is to destroy both. While it may be true that the line is greater than the sum of its points, it is nevertheless also true that there is no line without points.

Summary

1. Tonal music can be more unified than atonal music because of the presence of one or more common measures among its tones.

2. Tonal music can be more various than atonal music because the functional combinations and permutations available to the single tone are greater.

3. Tonal music can be more complexly ordered than can atonal music because the quantity of levels of hierarchical order can be greater.

4. Tonal music can be more consistent than atonal music because tonal music is a formalism the whole of which is patterned after the nature of its elements, and in which, therefore, the ordering of the macrocosm reflects the ordering of the microcosm.

5. Tonal music can be more expressive than can atonal music because tonal music can draw upon the expressive variety of its greater quantity of functional complexes, and thus it need not resort to the violation of cultural norms for its expression.

6. Tonal music can have greater continuity than can atonal music, because the continuity in tonal music is maintained at the deepest level by the presence of one or more common measures between the pitches, rather than by the relatively superficial methods of continuous flow, cyclic form, or serial order.

Appendix A

Glossary

THE FIRST definition for each of the following words comes from Webster's Dictionary. The second is my attempt to give a clearer subjective, and in some cases, objective, meaning to the word. The objective definition, where it appears, is not strictly speaking a definition at all, but is rather a description of the physical state of affairs which I believe to be the cause of the psychological phenomena indicated by the word.

CHORD

I. Agreement of musical sounds; accord, harmony. . . . A combination of tones which blend harmoniously when sounded together, because the pitch frequencies are in the ratios of small whole numbers.

II. A set of tones, which when sounded together, appear to constitute a model for, or variant of, the overtone series. A chord is thus a set of tones with a single perceptibly predominant root, corresponding to the fundamental of the overtone series. The ratio of the root strength to the consonance must be large enough that the compound wave pattern can be perceived as an entity. See Chapter III.

CONSONANCE

I. Agreement or congruity; harmony . . . a pleasing combination of tones; euphony . . . A combination of tones giving a sense of repose, that is, not demanding resolution; in contrast to dissonance. Acoustically this implies simple ratios between the vibration rates of the tones constituting the consonance.

II. The property of some sets of tones of sounding as though they fit together. The consonance of a set of tones is a function of two variables: (1) the position of the set relative to the fundamental determined by the set, and (2) the ratio of the least common multiple of the set to the greatest common divisor of the set. These two numerical factors can be combined in different ways to produce a coefficient of total consonance. The way I have chosen is:

(100 x property I) + property II.

See Chapter II.

FUNCTION

I. Webster's Dictionary gives no definition for "function" in the musical sense, but it gives one for "function" in a general sense:

Any quality, trait, or fact so related to another that it is dependent upon, and varies with, that other.

II. The function of a periodicity, whether tonal or metrical, is dependent upon its hierarchical position as a particular member of a perceptibly single-rooted set. See Chapters VIII and X.

KEY

I. A system, or family of tones, based on their relation to a keynote, or tonic, from which it is named . . .

II. The seven-tone major diatonic set or the seven-tone harmonic minor diatonic set, in any of the following four tunings or temperaments: Pythagorean, Just, Meantone, Equal-tempered. See Chapter VII.

METER

I. That part of rhythmical structure concerned with the division of a composition into measures by means of regularly occurring accents, each measure consisting of a uniform number of beats, or time units, of which the first has the strongest accent . . .

II. Meter is number mixed with time; or repetition of equal time intervals, or repetition of repetition of such time intervals, (etc.). See Chapter X.

MODE

I. An arrangement of the eight diatonic tones of an octave according to one of certain fixed schemes of their intervals; an octave species.

II. A diatonic set of tones, ordered in time in such a way as to emphasize a particular tone of the set. To each key there correspond seven different modes. See Chapter VII.

MODULATION

I. Act or process of changing from one key to another; a shifting of tonality so that the succeeding tones center upon a new keynote; . . .

II. Modulation is a change of the basic diatonic set. See Chapter IX.

ROOT

I. The tone from whose harmonics, or overtones, a chord is composed; often simply the lowest tone of a chord in its normal position.

II. That tone of a set of tones which is heard as being most important, apart from its duration, orchestration, spatial or temporal position relative to the other tones. The root of a set of tones is that tone whose integral representation in the set is a power of 2. If the set contains more than one such tone, the one with the lowest power of 2 is the root. Root strength diminishes rapidly as the power of 2 increases. See Chapter III.

TONALITY

I. The principle of key in music, the affinity of a group or series of tones for a central tone or tonic; the character which a composition has by virtue of its key, or through the family relationship of all its tones or chords to the keynote, or tonic of the whole.

The predominance of the tonic as the link which connects all the tones of a piece, we may, with Fetis, term the principle of tonality.

—Helmholtz

II. If music is tonal, then people who listen to it can sing, relative to a given passage, a tone which they feel to be the most important tone. Music is tonal if, and only if, the majority of its adjacent tones, whether occurring simultaneously or consecutively, constitute sets having single predominant roots. See Chapter XI.

TONIC

I. Of or pertaining to the keynote; . . .

II. The predominant root of a key. See Chapter VII.

Appendix B

Hindemith

PAUL HINDEMITH has developed the idea of musical hierarchy in his *Craft of Musical Composition,* Vol. I.[1] The forest sighted in this book is similar to the one which is being explored here. But in that book trees are described as growing in that forest which do not and could not grow there. Specifically:

Hindemith speaks of non-harmonic tones, although he has accepted all sets of tones as chords. He is concerned with the spreading out of chords in time. At the same time he distinguishes between harmonic and non-harmonic tones, although he has removed any objective basis for the distinction between the two by his use of the word "chord". Thus the only basis for making the distinction is the listener's intuition. This is not theory.

He claims to have derived a twelve-tone scale from a single tone by a particular method, whereas in fact he did not follow the method, for if he had, the scale would have had many more than twelve tones. He arbitrarily selects some of these tones and rejects others of them.

He assumes that a single root will predominate in any set no matter how many other conflicting roots are present. As a result of this assumption he directs his attention and the main body of his chordal theory to distinguishing between the indistinguishable, or barely distinguishable, while at the same time he lumps the easily distinguishable into one category. Metaphorically speaking his theory of chordal tensions gives careful gradation to the different degrees of brown, but groups the primary colors into one or two categories.

His Series II gives unequivocal roots to some intervals which, according to this book, either have no roots or conceptually equivocal ones. These intervals are the minor third, the major second,

[1] Hindemith, Paul, *The Craft of Musical Composition,* Vol. 1, Associated Music Publishers, N.Y., 1937.

and the minor second, together with their inversions. His root theory is based upon relationships between subsidiary tones, rather than on the relationships between fundamentals.

These errors are not superficial, but lie at the basis of the theory, thus influencing the super-structure. It is sometimes said that we should ignore the theoretical weaknesses of Hindemith's theory and attend to its "practical" application. The theory, according to this, is true because it works; that is, it enables us to make sense out of the music we apply it to. To this it can be answered that from a contradiction, explicit or implicit, we can prove anything, even the presence of order where there is none.

Despite these objections, I regard *The Craft of Musical Composition,* Vol. I, as one of the important books of the century, for it reasserts the relevance of the overtone series to music, and reopens the path to the connection between science and art which is so desperately needed today.

Appendix C

Euler

LEONHARD EULER, in his *Tentamen Novae Theoriae Musicae* proposed a measure of "sweetness" which depended solely on Property II.[1] He did not call it consonance, reserving for that word the meaning: sounding with. Thus any set of tones for him was consonant if they were sounded at the same time. But he defines "sweetness" as follows:

> Henceforth we shall call the least common multiple of the simple sounds constituting the consonance the *exponent* of the consonance. The manner of finding the degree of sweetness when this exponent is given is . . . as follows: When the exponent is resolved into all its simple sounds, let the sum of these be s. Let the number of these factors be n; the degree of sweetness proposed will be
>
> $$s - n + 1;$$
>
> thus, by so much less is this number found to be, by that much the consonance will be sweeter, or easier to perceive.[1]

This measure was not accepted by many theorists because it gave the major second, the minor third, and the minor sixth the same degree of sweetness. Instead of improving on the measure, theorists abandoned the whole idea of deriving a mathematical measure which would agree with the musician's intuitions.

Euler also suggested that the dominant seventh chord could be represented by the numbers 4:5:6:7.[2]

[1] Euler Leonhard, *Tentamen Novae Theoriae Musicae*, Chapter IV, Section 6, *Opera Omnia*, Series 3, Vol. I, Lipsiae and Berolini, 1926.

[2] Euler, Leonhard, *Du Véritable Caractère de la Musique Moderne*, Ibid.

Bibliography

Aristotle, *Problems I-XXI,* Harvard University Press, Cambridge, 1936, 461 pp.

Culver, Charles, *Musical Acoustics,* McGraw-Hill, N.Y., 1956.

Euler, Leonhard, *Letters to a German Princess,* Vol. I, Harpers, N.Y., 1833, 386 pp.

Euler, Leonhard, *Tentamen Novae Theoriae Musicae,* Opera Omnia, Series III, Vol. I, Lipsaiae and Berolini, 1926.

Fétis, F. J., *Traité de L'Harmonie,* Brandus et Cie, Paris, 1853, 275 pp.

Heisenberg, *Physics and Philosophy,* Harpers, N.Y., 1958, 206 pp.

Helmholtz, Hermann, *Sensations of Tone,* Longmans Green, London 1885.

Hesse, Hermann, *Magister Ludi,* Henry Holt, N.Y., 1949, 502 pp.

Hiller, Lejaren, and Isaacson, Leonard, *Experimental Music,* McGraw Hill, 1959, 197 pp.

Hindemith, Paul, *A Composer's World,* Harvard University Press, Cambridge, 1952, 221 pp.

Hindemith, Paul, *The Craft of Musical Composition,* Associated Music Publishers, N.Y., 1937, 223 pp.

Jarrell, Randall, *Poetry and the Age,* Knopf, N.Y., 1953, 246 pp.

Kent, Charles, *Tonal Expansion in the Early Romantic Period,* University of Rochester Press, N.Y., 1954, thesis.

Lambert, Constant, *Music Ho,* Faber and Faber, London, 1937.

Lawrence, David Herbert, *Studies in Classic American Literature,* Doubleday Anchor Books, N.Y., 1953, 191 pp.

Leibnitz, Gottfried Wilhelm, *Philosophical Papers and Letters,* University of Chicago Press, Chicago, 1956, 1228 pp.

Lynch, *Christ and Apollo,* Sheed and Ward, N.Y., 1960, 267 pp.

Mann, Thomas, *Doctor Faustus,* Alfred A. Knopf, N. Y., 1948, 510 pp.

May, Rollo, *Existence,* Basic Books, N.Y., 1960, 445 pp.

McHose, A. I., *Basic Principles of the Technique of the Eighteenth and Nineteenth Century Composition,* Appleton-Century-Crofts, N.Y., 1951, 399 pp.

Plato, *Dialogues of Plato,* Vol. I., Cratylus, Phaedrus, Macmillan, N.Y., 1892, 594 pp.
Pleasants, Henry, *The Agony of Modern Music,* Simon and Schuster, N.Y., 1955, 180 pp.
Piston, Walter, *Harmony,* Norton, N.Y., 1948, 334 pp.
Redfield, John, *Music, A Science and an Art,* Alfred Knopf, N.Y., 1928, 307 pp.
Schenker, Heinrich, *Harmony,* Chicago University Press, Chicago, 1954, 359 pp.
Schoenberg, Arnold, *Theory of Harmony,* Philosophical Library, N.Y., 1948, 336 pp.
Schroedinger, Erwin, *What is Life?,* Macmillan, N.Y., 1957, 91 pp.
Shirlaw, Matthew, *The Theory of Harmony,* Novello and Co., London, 484 pp.
Teilhard de Chardin, Pierre, *The Phenomenon of Man,* Harpers, N.Y., 1959, 318 pp.
Thompson, W. E. *A Clarification of the Tonality Concept,* Unpublished Thesis, 1952, 219 pp.
Valéry, Paul, The Course in Poetics, First Lesson, in *The Creative Process,* Ed. by Brewster Ghiselin, Mentor, N.Y., 1955, 251 pp.
Weil, Simone, *The Need for Roots,* Putnam, N.Y., 1952, 302 pp.
Yasser, Joseph, *A Theology of Evolving Tonality,* Am. Lib. of Musicology, N.Y., 1932, 381 pp.
Zuckerkandl, Victor, *Sound and Symbol,* Pantheon, N.Y., 1956, 399 pp.